# I. Unstructured Search

| | | | | | | |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | ... | 1 | ... | 0 |
| 0 | 1 | 2 | ... | w | ... | $2^n-1$ |

Let $N=2^n$.

we are looking for "w" in the list.

1. Using a classical computer, we can find such an element in $O(N)$ time. ($O(N) = O(2^n)$)
2. Using a quantum computer, the Grover's algorithm find such an element in $O(\sqrt{N})$ time.

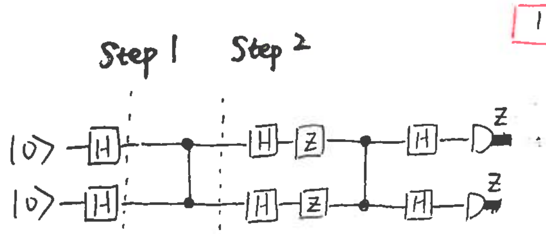
# II Oracle Function

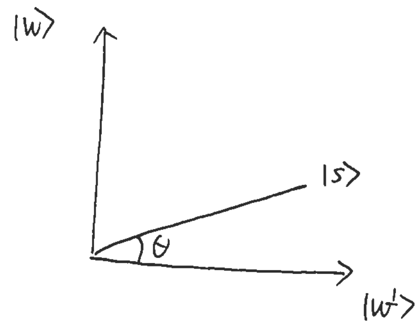
$$f(x) = \begin{cases} 0 & x = w \\ 1 & x \neq w \end{cases}$$

Oracle answers a question, but it's not necessarily clear how to implement it. You can think of it as a blackbox

**Example** $w = 11$

| $x$ | $f(x)$ |
|---|---|
| 00 | 0 |
| 01 | 0 |
| 10 | 0 |
| 11 | 1 |




**Step 1:** $H \otimes H |00\rangle = \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)$

$$= \frac{1}{2}|00\rangle + \frac{1}{2}\left(|01\rangle + |10\rangle + |11\rangle\right)$$

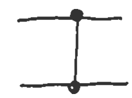
$$\therefore \ |s\rangle = \frac{1}{2}|w\rangle + \frac{\sqrt{3}}{2}|w^\perp\rangle, \quad |w^\perp\rangle = \frac{1}{\sqrt{3}}\left(|01\rangle + |10\rangle + |11\rangle\right)$$

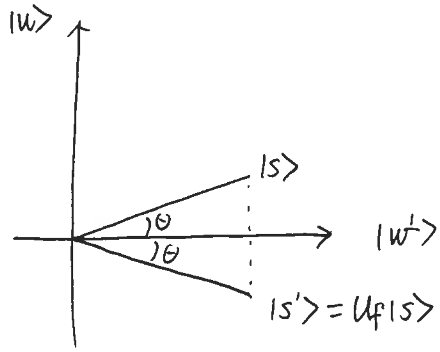
$$\cos\theta = \frac{\sqrt{3}}{2}, \ \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}$$

**Step 2:** Oracle function $U_f|x\rangle = (-1)^{f(x)}|x\rangle$. Reflection about $|w^\perp\rangle$

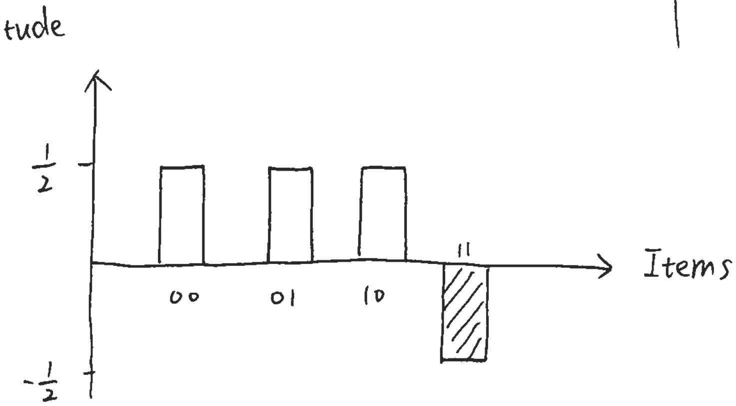
$U_f|00\rangle = |00\rangle$ $\quad U_f|01\rangle = |01\rangle$ $\quad U_f|10\rangle = |10\rangle$ $\quad U_f|11\rangle = -|11\rangle$

$$U_f = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} =: CZ$$

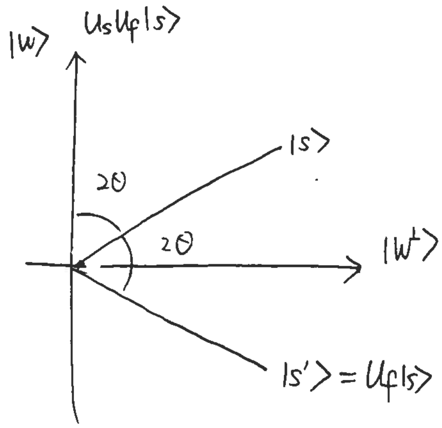




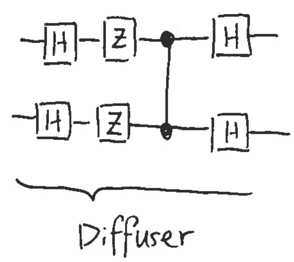

$\theta = \frac{\pi}{6}$

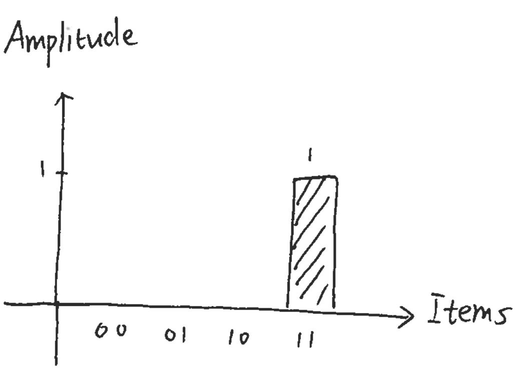



**Step 3:** $U_s = 2|s\rangle\langle s| - I$ $\quad$ Reflection about $|s\rangle$.

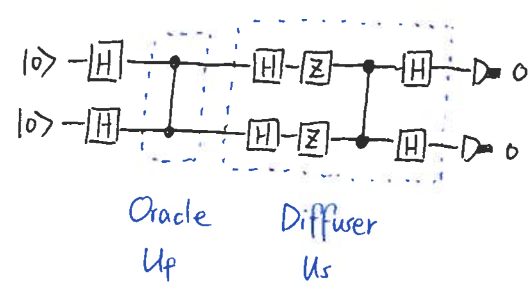



$3\theta = \frac{\pi}{2}$.




The Final Circuit for Solving the Two-Qubit Grover's Algorithm



## Exercises :

1. What if $|w\rangle = |00\rangle$ ?

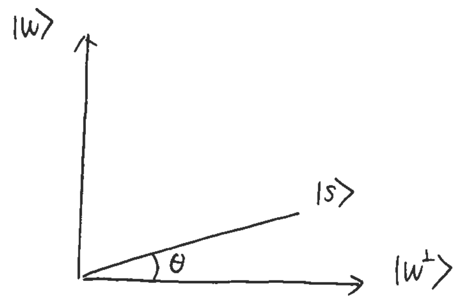
2. What if $|w\rangle = |01\rangle$ ?

## III Summary of Grover's Algorithm

Step 1: $H^{\otimes n}|0\rangle^{\otimes n} = |s\rangle = \frac{1}{\sqrt{N}}\sum_{x=0}^{N-1}|x\rangle$

Putting the qubits into every different possible input $x$

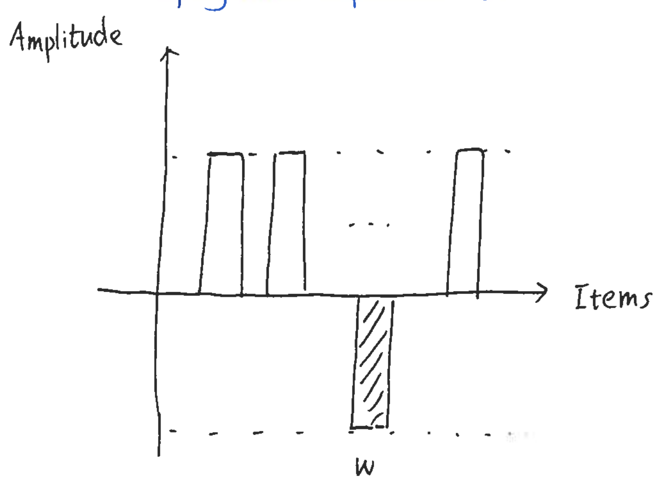
$|s\rangle = \frac{1}{\sqrt{N}}|w\rangle + \frac{\sqrt{N-1}}{\sqrt{N}}\sum_{x\neq w}|x\rangle$ Denote $|w^{\perp}\rangle = \frac{1}{\sqrt{N-1}}\sum_{x\neq w}|x\rangle$



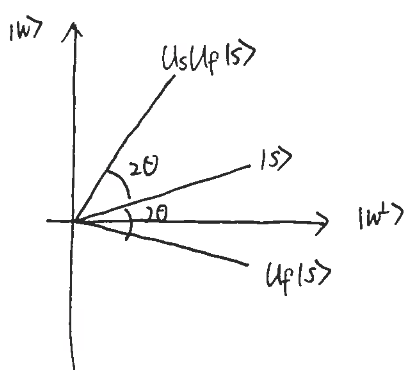

$\cos\theta = \frac{\sqrt{N-1}}{\sqrt{N}}$ $\sin\theta = \frac{1}{\sqrt{N}}$ $\theta = \arcsin\frac{1}{\sqrt{N}}$

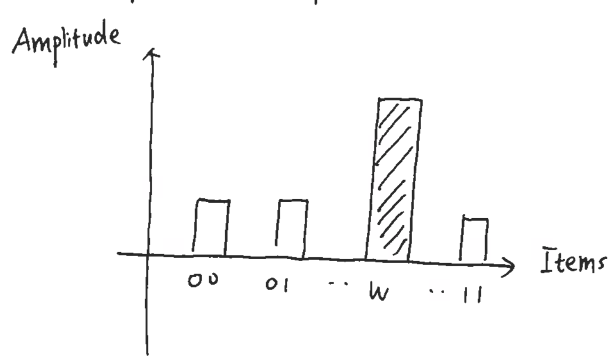
Step 2: $U_f|x\rangle = (-1)^{f(x)}|x\rangle$ Phase inversion $U_f = I - 2|w\rangle\langle w|$

Flipping the amplitude of JUST the item that we are looking for.




Step 3: $U_s = 2|s\rangle\langle s| - I$ ➡ Step 2&3: Amplify the amplitude of the marked item $w$



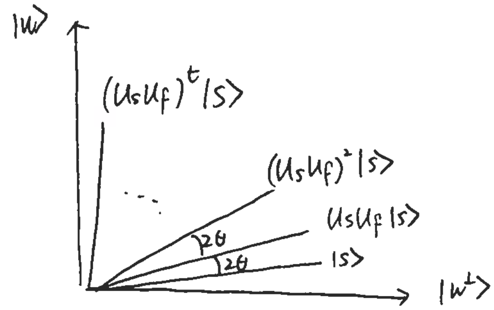



Finally $|\psi_t\rangle = (U_s U_f)^t |\psi_0\rangle$

$t$ amount of times

initial state

final state




Compute the complexity :

$(2t+1)\theta \approx \frac{\pi}{2}$ $\qquad \theta = \arcsin \frac{1}{\sqrt{N}}$

$\therefore \; t = \left(\frac{\pi}{2\theta} - 1\right)\frac{1}{2} = \frac{\pi}{4\theta} - \frac{1}{2}$

Morewer $\lim\limits_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$ , so $\sin\theta \approx \theta$

$\therefore \; t = \frac{\pi}{4\theta} - \frac{1}{2} \approx \frac{\pi}{4\sin\theta} - \frac{1}{2} = \frac{\pi}{4\sin\sin^{-1}\frac{1}{\sqrt{N}}} - \frac{1}{2} = \frac{\pi}{4}\sqrt{N} - \frac{1}{2} = O(\sqrt{N})$