Quantum Error Correction and Fault Tolerance: Surface Codes

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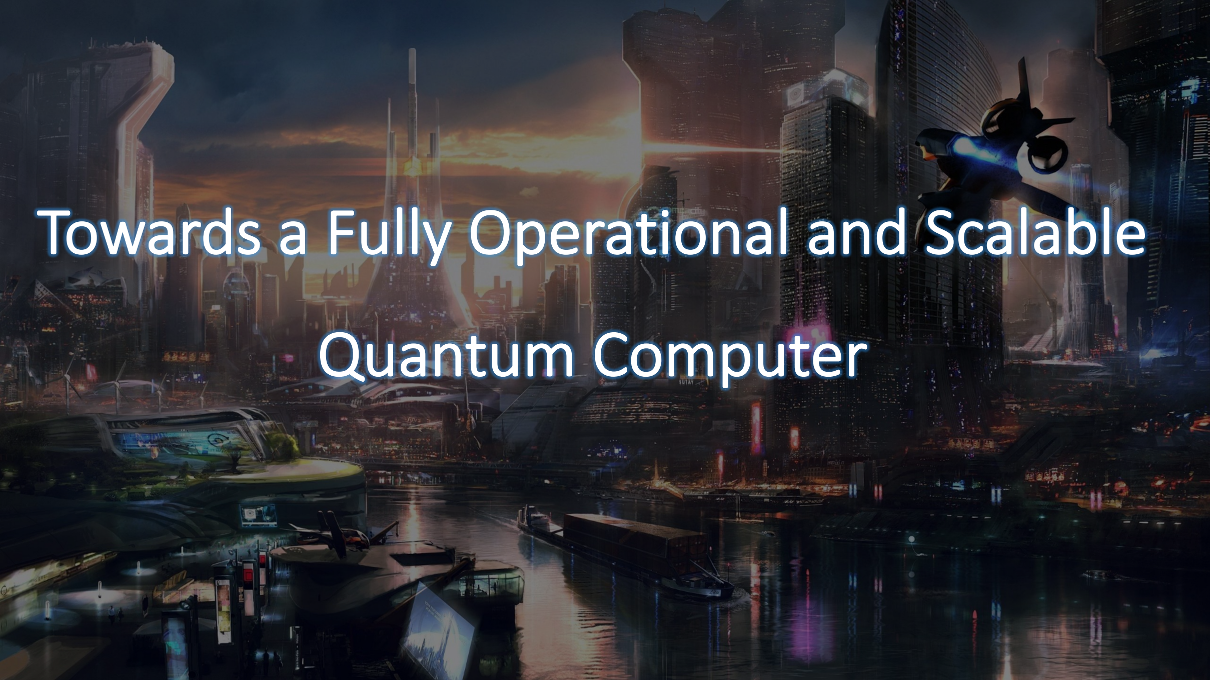

Towards a Fully Operational and Scalable Quantum Computer

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- Present in the transmission, processing, or storage of quantum information.

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- **Understand environmental decoherence processes and model them properly.**
- **Use error correction to protect quantum information against decoherence.**

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Binary Symmetric Channel (BCS_p)

IN: $b \in \{0, 1\}$

OUT: $\begin{cases} b, & \text{prob. } 1 - p \\ \neg b, & \text{prob. } p \end{cases}$

Assume $p \in [0, 1]$, the channel behaves independently for each bit that passes through it.

Examples of Single-Qubit Errors

Bit Flip X : $X|0\rangle = |1\rangle, X|1\rangle = |0\rangle$.

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Phase Flip Z : $Z|0\rangle = |0\rangle, Z|1\rangle = -|1\rangle$.

$$Y = iXZ = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}.$$

Complete Dephasing : $\rho \longrightarrow 1/2(\rho + Z\rho Z^\dagger)$.

Rotation R_θ : $R_\theta|0\rangle = |0\rangle, R_\theta|1\rangle = e^{i\theta}|1\rangle$.

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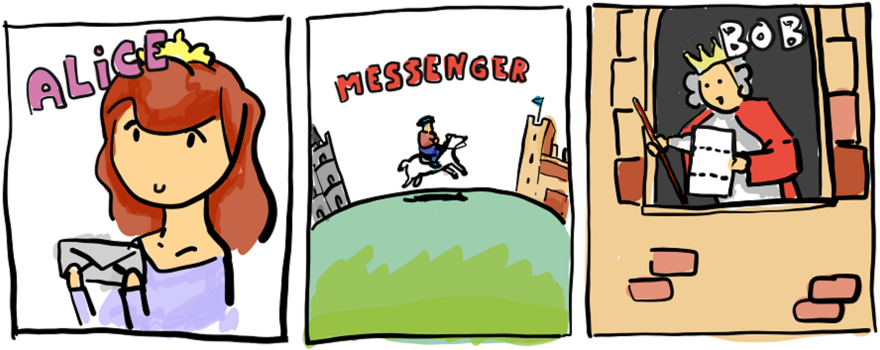
Definition

A single-qubit Pauli error could be one of the following single-qubit errors:

- A bit-flip error X;
- A phase-flip error Z;
- Both a bit-flip and a phase-flip error: Y.



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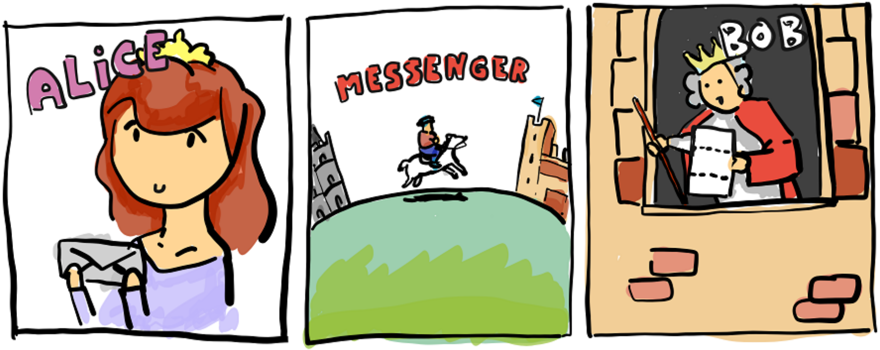

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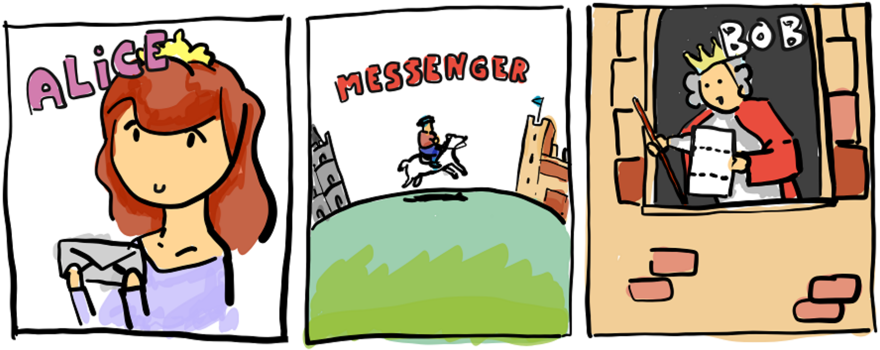

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1. Get a better communication channel (*BCS* with a smaller p).
2. **Use quantum error correction codes.**

3-Bit Repetition Code

1. Alice encodes $b \in \{0, 1\}$ as bbb, and sends the three bits through the channel.

$$0 \longrightarrow 000 \qquad 1 \longrightarrow 111.$$

2. Bob decodes the three bits he receives by taking the majority count.
The three bits that Bob receives may not be the same.

$$000 \longrightarrow \begin{cases} 100, \text{The 1st bit is flipped;} \\ 010, \text{The 2nd bit is flipped;} \\ 001, \text{The 3rd bit is flipped.} \end{cases} \qquad 111 \longrightarrow \begin{cases} 011, \text{The 1st bit is flipped;} \\ 101, \text{The 2nd bit is flipped;} \\ 110, \text{The 3rd bit is flipped.} \end{cases}$$

3. If no more than one bit is flipped, this method succeeds because flipping one bit does not change the majority.

Obstacles in Quantum Error Correction

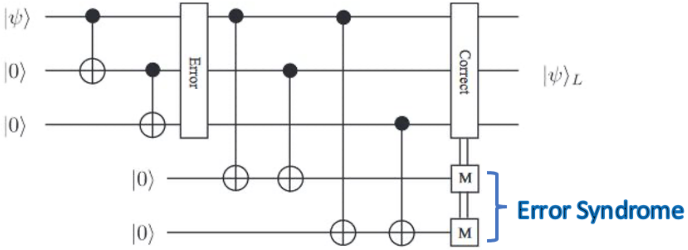
- No-cloning theorem forbids the classical repetition strategy.
- Measuring qubits to identify errors would collapse superpositions.
- Need to correct bit flip and phase errors.
- Need to handle continuous rotations, decohering maps, etc.
- Need to correct errors on multiple qubits.

[1] Daniel Gottesman: Quantum Error Correction and Fault Tolerance (Part 1) - CSSQI 2012.

Correct a Bit Flip Error

To correct a single bit flip error, we can encode the data as:

$$0 \longrightarrow 000, 1 \longrightarrow 111$$



If there is a single bit flip error, we can correct the state by choosing the majority of the three bits.

Analysis

State that Bob receives	Probability	Syndrome	Correction
$(\alpha \|100\rangle + \beta \|011\rangle) \|11\rangle$	$p(1-p)^2$	11	Flip the 1st qubit
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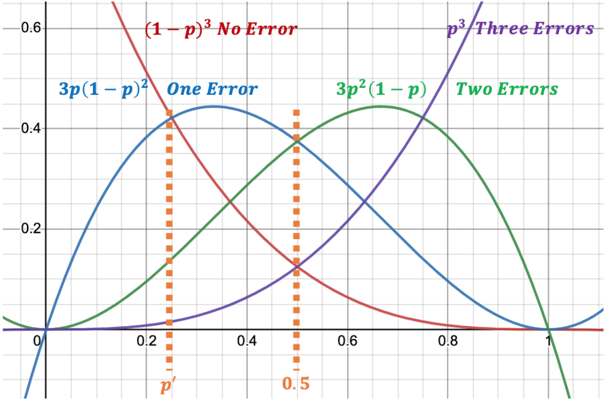
Suppose Bob measures 11, then he must either receives

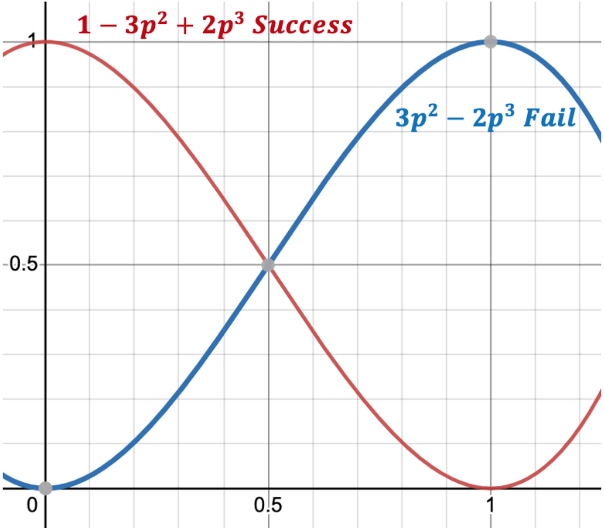
- $\alpha |100\rangle + \beta |011\rangle$ with probability $p(1-p)^2$, or
- $\alpha |011\rangle + \beta |100\rangle$ with probability $p^2(1-p)$.

When errors are rare, one error is more likely than two errors.



Repetition Code Corrects up to One Error

Error	Probability	Success/Fail
0	$(1-p)^3$	$1-3p^2+2p^3$
1	$3p(1-p)^2$	
2	$3p^2(1-p)$	$3p^2-2p^3$
3	$(1-p)^3$	



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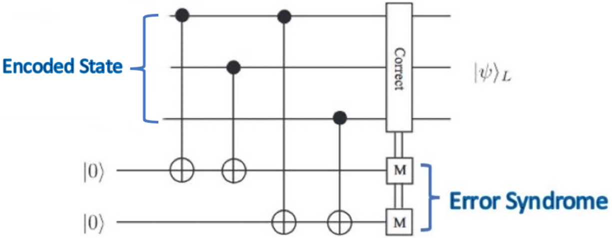
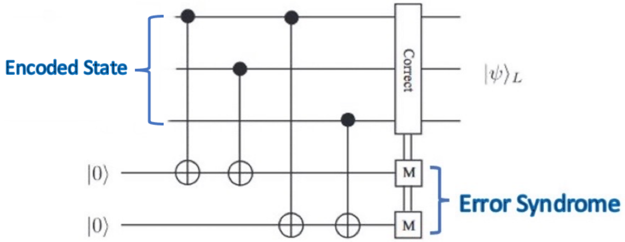
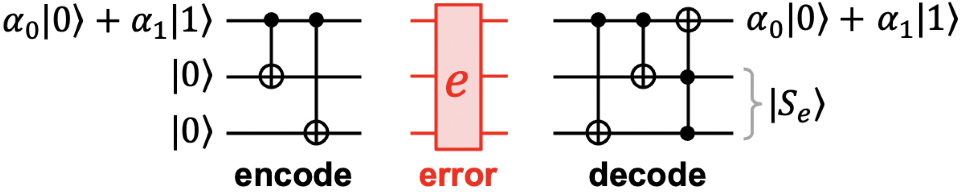
Detect the error: Measure the error, not the data.







[1]Cleve, R. (2021). Introduction to Quantum Information Processing. Retrieved June 6, 2023.

Correct a Phase Flip Error

Can we use the same procedure to correct a single-qubit phase flip error?

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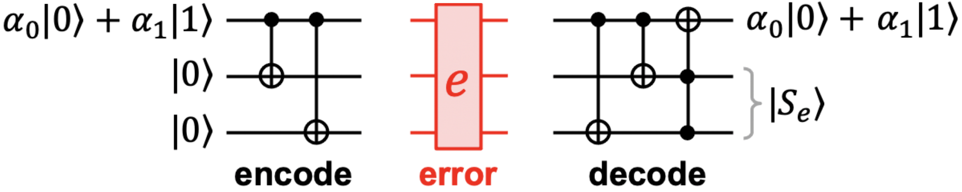

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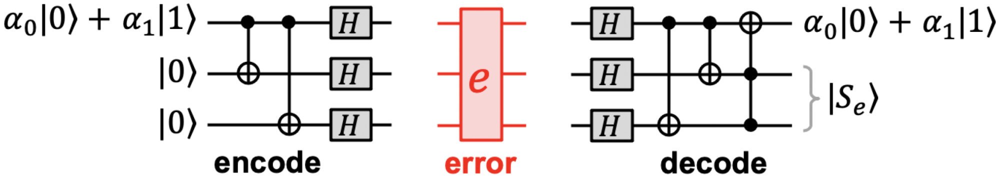
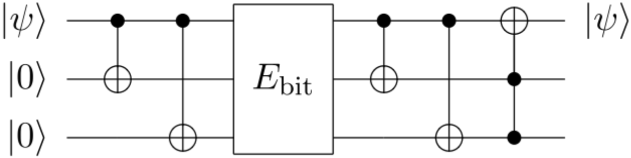
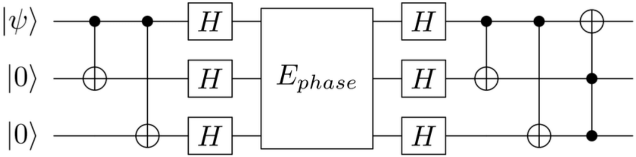


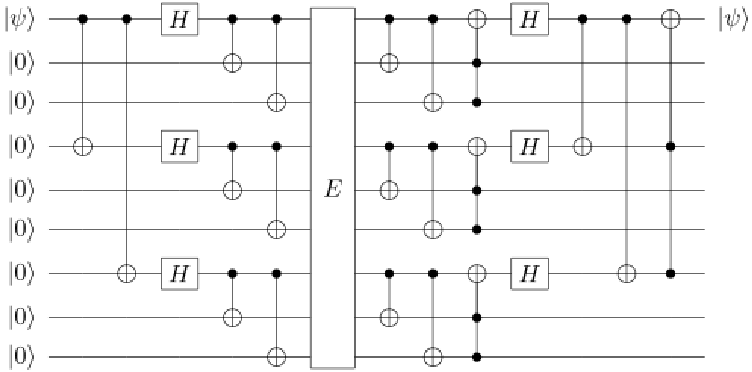
Correct a Phase Flip Error

Since $HZH = X$, we can reduce the problem of the phase flip error correction to an instance of the bit flip error correction.





Correct a Single-Qubit Pauli Error



Shor's Nine-Qubit Code

Stabilizer Formalism

Gottesman, D. (1997). Stabilizer codes and quantum error correction. California Institute of Technology.

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$$|\bar{\psi}\rangle = \alpha\,|\bar{0}\rangle + \beta\,|\bar{1}\rangle = \alpha\,|000\rangle + \beta\,|111\rangle\,, \quad \alpha, \beta \in \mathbb{C}, \quad |\alpha|^2 + |\beta|^2 = 1.$$

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$$\bar{X}_i \bar{Z}_i = -\bar{Z}_i \bar{X}_i, \quad 1 \leq i \leq k$$
$$\bar{X}_i \bar{Z}_j = \bar{Z}_j \bar{X}_i, \quad 1 \leq i, j \leq k, \quad i \neq j$$
$$\bar{Z} |\bar{0}\rangle = |\bar{0}\rangle, \quad \bar{Z} |\bar{1}\rangle = - |\bar{1}\rangle$$
$$\bar{X} |\bar{+}\rangle = |\bar{+}\rangle, \quad \bar{X} |\bar{-}\rangle = - |\bar{-}\rangle$$

Stabilizer Code

Consider three groups of Pauli operators.

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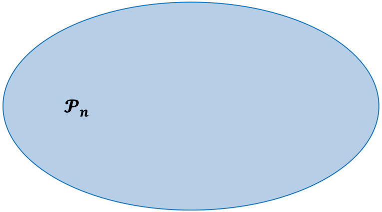

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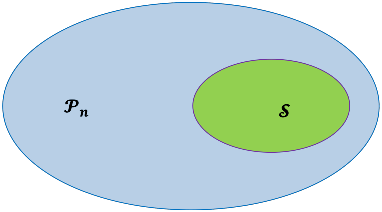
2. Stabilizer group: $\mathcal{S} = \langle M_1, M_2, \ldots, M_{n-k}\rangle$, $-I \notin \mathcal{S}$. $\mathcal{S} \subset \mathcal{P}_n$. $\mathcal{S}$ Abelian.



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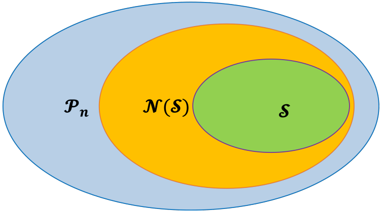

Definition

Stabilizer codes are a class of quantum error-correcting codes. Its code space $\mathcal{C}$ is the joint $+1$ eigenspace of $\mathcal{S}$.

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$|\overline{\psi}\rangle$ is called a *codeword* in $\mathcal{C}$, where

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Example

Consider $\mathcal{S} = \langle XX, ZZ \rangle$ on two qubits. Then $\mathcal{C} = \left\{ \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \right\}$.

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  They are anticommuting Pauli pairs acting non-trivially on $|\overline{\psi}\rangle$.
- All other operators in $\mathcal{P}_n$ anti-commute with at least one element in $\mathcal{S}$ and map a codeword $|\overline{\psi}\rangle$ onto a state outside the code space $\mathcal{C}$.

Fundamental Theorem of Stabilizer Theory

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- More generally, we think of non-maximal stabiliser groups as a description for the embedding of $k = n - m$ "logical" qubits into a space of n "physical" qubits.

Example: Four-qubit code $[[4, 2, 2]]$

$$\mathcal{S} = \langle XXXX, ZZZZ \rangle$$

- What is the dimension of the code space?

Code Distance

Definition

Let d be the distance of a stabilizer code $\mathcal{C}(\mathcal{S})$, $|P|$ denotes the weight of $P \in \mathcal{P}_n$, the number of physical qubits on which P acts nontrivially. Then

$$d := \min_{P \in \mathcal{N}(\mathcal{S})/\mathcal{S}} |P|.$$

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Example: Four-qubit code $[[4, 2, 2]]$

$$\mathcal{S} = \langle XXXX, ZZZZ \rangle$$

- Find pairs of mutually anti-commuting Paulis which commute with $XXXX$, $ZZZZ$.
- What is the code distance?

Detectable Errors

Definition

$\forall A, B \in \mathcal{P}_n,$

$$[A, B] = AB - BA \qquad \{A, B\} = AB + BA.$$

$[A, B] = 0$, $\{A, B\} = 0$ denote when A and B commute, anticommute respectively.

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E is an undetectable error: When $E \in \mathcal{N}(\mathcal{S})$, $[E, M] = 0$, $\forall M \in \mathcal{S}$.

- $E \in \mathcal{S}$, E is trivial.
- $E \in \mathcal{N}(\mathcal{S}) \setminus \mathcal{S}$, E is a logical operator. **BAD!**
- ∗ E maps a codeword to another codeword.

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  - E maps a codeword to another codeword.

E is an detectable error: When $E \notin \mathcal{N}(\mathcal{S})$, $\exists M \in \mathcal{S}$ s.t. $\{E, M\} = 0$.

Detectable Errors Cont.

Lemma

A stabilizer code of distance d can detect all Pauli errors of weight less than d (as long as they are not elements in S).

When a Pauli error has weight greater than or equal to d, it may or may not be detected.

Detectable Errors Cont.

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`Two questions to think about:`

Detectable Errors Cont.

Lemma

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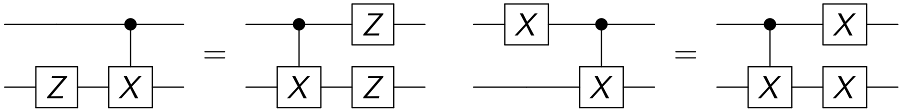
Two questions to think about:

- E and F have the same error syndrome iff $E^\dagger F \in \mathcal{N}(\mathcal{S})$.
- A code of distance $d = 2t + 1$ can correct any error of weight t.

Fault-tolerant Technique: Transversality



[3]Gottesman, D. (2000). Fault-tolerant quantum computation with local gates. Journal of Modern Optics, 47(2-3), 333-345.

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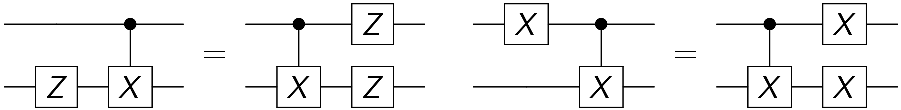

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A transversal logical operator is **NOT** implemented by any multi-qubit physical operation acting on the same code block.

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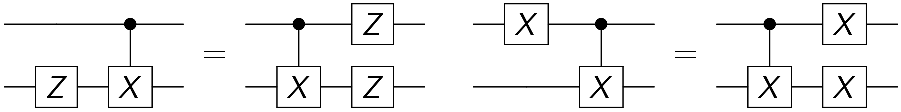

Definition

A transversal logical operator is **NOT** implemented by any multi-qubit physical operation acting on the same code block.

- Transversality prevents any errors from spreading within a block, so a single physical error cannot cause a whole block of codes to go bad.

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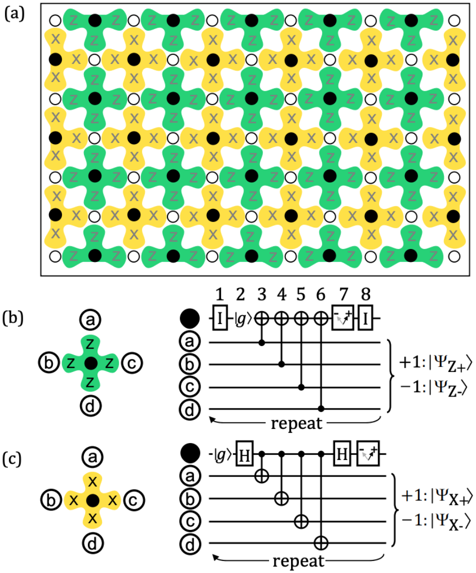
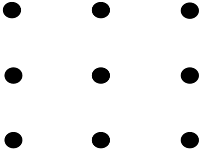


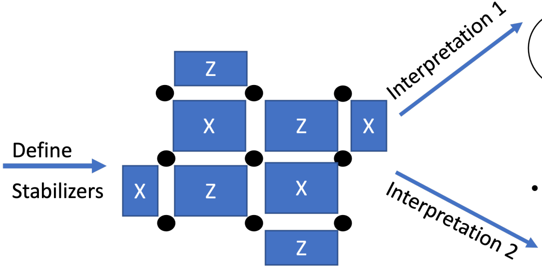
2D Surface Code

- A family of stabilizer codes defined on a 2D lattice of qubits.
- Pros: high error threshold and the planar layout of physical qubits. Each physical qubit only interacts with its nearest neighbours.
- Cons: the available transversal logical gates are limited.

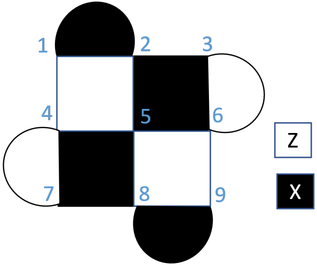
[d]Bravyi, S. B. & Kitaev, A. Y. (1998). Quantum codes on a lattice with boundary. arXiv preprint quant-ph/9811052



A square array of qubits



Local stabilizers are defined all over the lattice.



- Independent stabilizers: $m = 8$

$$Z_3Z_6, \quad Z_1Z_2Z_4Z_5$$
$$Z_8Z_9, \quad Z_5Z_6Z_8Z_9$$
$$X_1X_2, \quad X_2X_3X_5X_6$$
$$X_8X_9, \quad X_4X_5X_7X_8$$

- Logical operators: $\mathrm{k} = 1$

$$\bar{X} = X_1X_2X_3, \quad \bar{Z} = Z_1Z_2Z_3$$

- Code distance: $\mathrm{d} = 1$

The Smallest Interesting Surface Code.

Welcome to the error correction zoo

EC ZOO BETA

Jump to ▸ Linear binary, Additive q-ary, RS, RM, LDPC, Polar, Rank-metric, STC, Stabilizer, CSS, Good QLDPC, Kitaev surface, Color, Topological, Holographic, EAQECC, GKP, Cat

Classical Domain ▸ Binary Kingdom, Galois-field Kingdom, Matrix Kingdom, Lattice Kingdom, Spherical Kingdom, Ring Kingdom, Group Kingdom
Quantum Domain ▸ Qubit Kingdom, Modular-qudit Kingdom, Galois-qudit Kingdom, Bosonic Kingdom, Fermionic Kingdom, Spin Kingdom, Group Kingdom, Category Kingdom

Code lists ▸ Approximate quantum codes, Binary linear codes, Quantum CSS codes, Codes with notable decoders, Dynamically generated quantum codes, Asymptotically good QLDPC codes, Hamiltonian-based codes, Holographic codes, Quantum codes based on homological products, LDPC codes, MDS codes, Perfect codes, q-ary linear codes, Quantum LDPC codes, Quantum codes with code capacity thresholds, Quantum codes with fault-tolerant gadgets … (see all)

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Code lists
All codes
Glossary of concepts

css

More

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Binary linear code defined on edges on a regular graph G such that each subsequence of bits corresponding to edges in the neighborhood any vertex belong to some \textit{short} binary linear code C_0. Expansion properties of the underlying graph can yield efficient decoding algorithms. More …

Stats at a glance: **275** code entries, **15** kingdoms, **2** domains, **72** classical codes, **124** quantum codes, **79** abstract property codes, **27** topological codes, **33** CSS codes, **44** quantum LDPC codes, and **26** bosonic codes.

Fault-tolerant Protocols for Surface Codes

Measurement-based schemes for performing logical operations in surface code.

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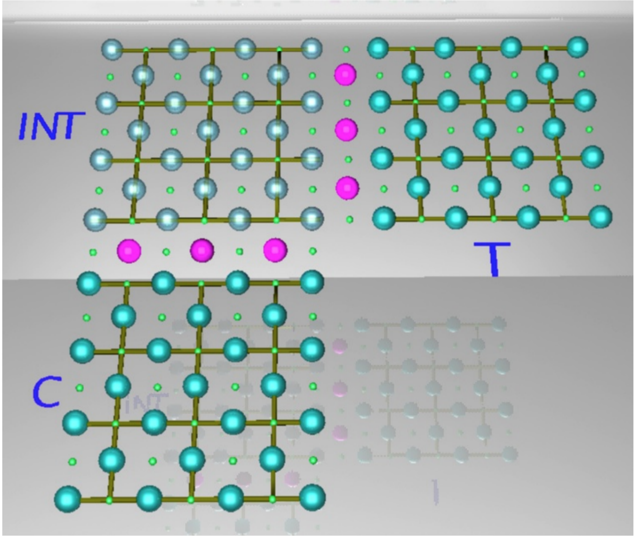
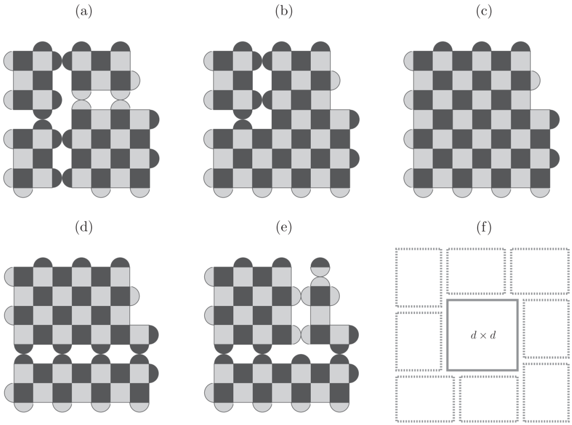
Magic state distillation: implement the non-Clifford T gate.

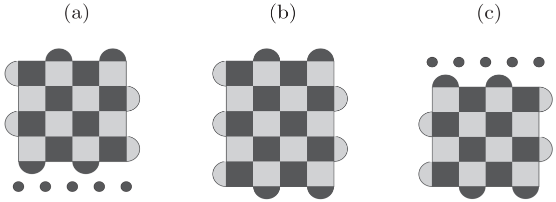


Lattice Surgery

- Three surface code patches to perform a lattice surgery for a fault-tolerant implementation of the logical CNOT gate.
- Control (C) and target (T) surfaces interact by merging and splitting with the intermediate surface (INT).

[e] Horsman, C., Fowler, A. G., Devitt, S. & Van Meter, R. (2012). Surface code quantum computing by lattice surgery. New Journal of Physics, 14(12), 123011.



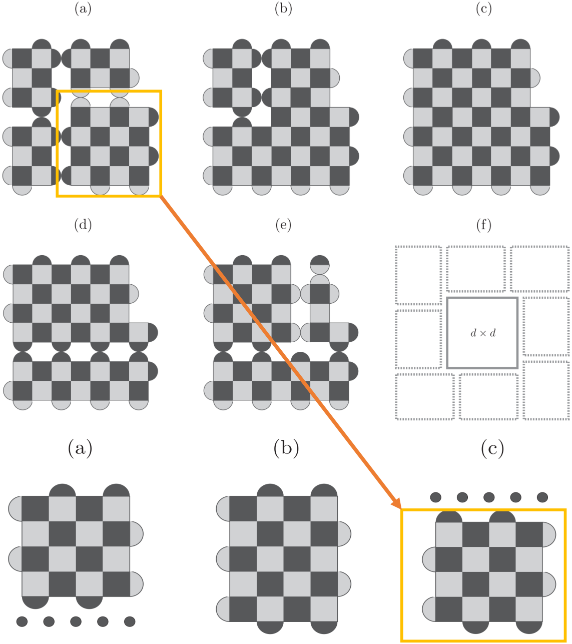


Code Deformation

- Fault-tolerant procedure for rotating a surface code by 90 degrees and reflecting it about the x axis.
- Realizing a logical H gate.

[e] Bombín, H. & Martin-Delgado, M. A. (2009). Quantum measurements and gates by code deformation. Journal of Physics A: Mathematical and Theoretical, 42(9), 095302.

[f] Vuillot, C., Lao, L., Criger, B., Almudéver, C. G., Bertels, K. \& Terhal, B. M. (2019). Code deformation and lattice surgery are gauge fixing. New Journal of Physics, 21(3), 033028.



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Gauge Fixing

- Different gauge fixing operations result in different stabilizer groups.
- This method is used to switch between the Steane code and the quantum Reed-Muller code.

$$\mathbb{S}_S \xrightarrow{\mathbb{M}\left(L_g^Z\right)} \mathbb{S}_4 = \mathbb{S}_S \cup L_g^Z$$

$$\mathbb{S}_S \xrightarrow{\mathbb{M}\left(L_g^X\right)} \mathbb{S}_E = \mathbb{S}_S \cup L_g^X$$

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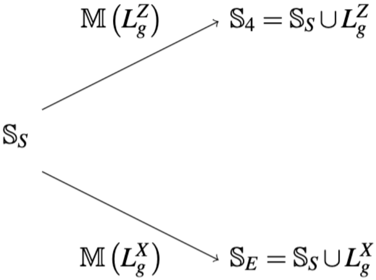

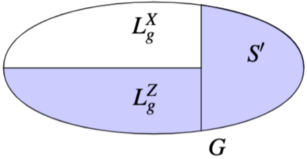

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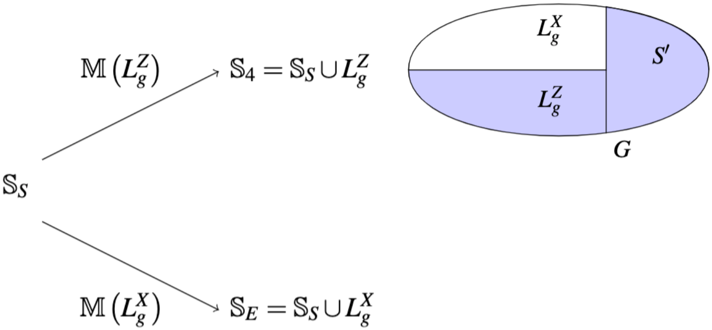

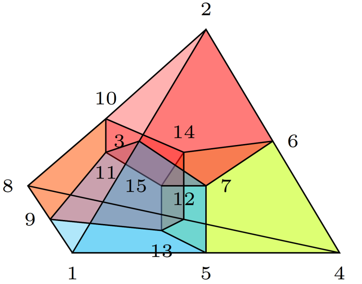

15-qubit Quantum Reed-Muller Code

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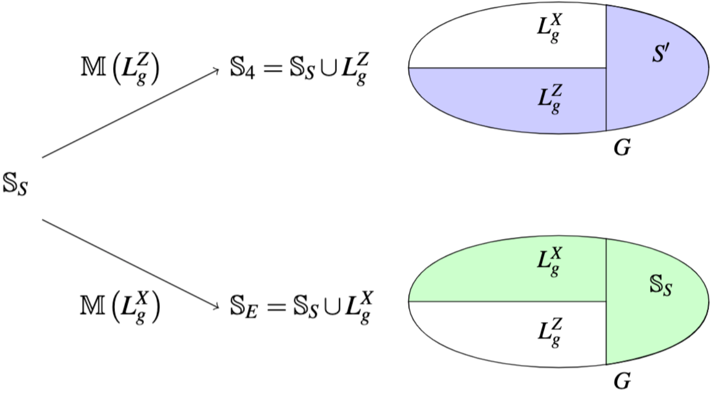

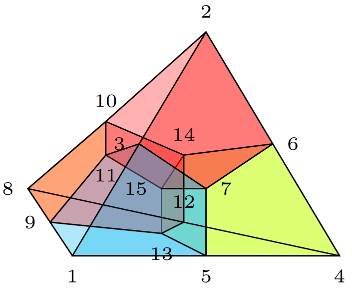

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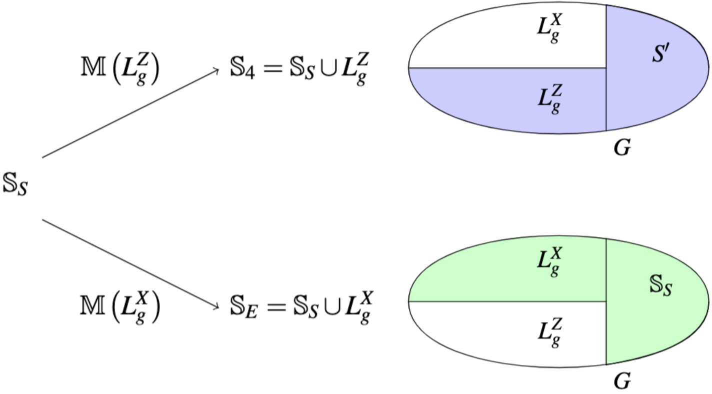

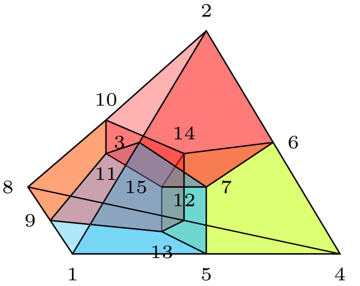

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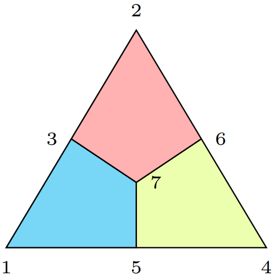
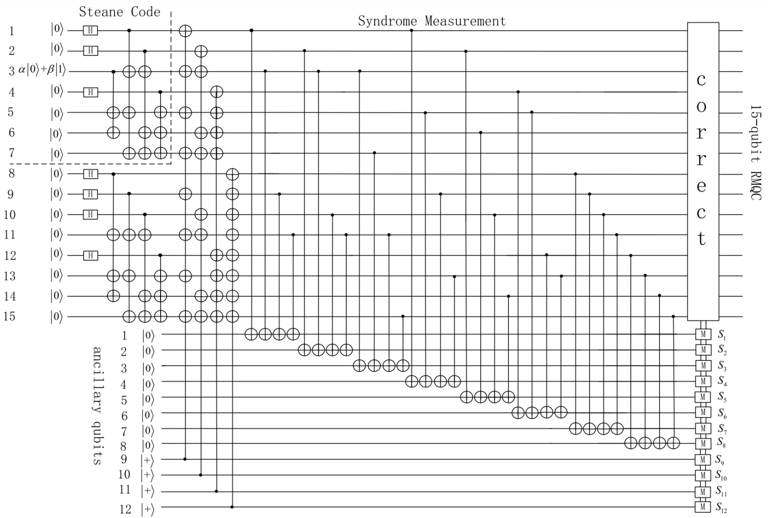


7-qubit Steane Code

Switch between Steane Code and Quantum Reed-Muller Codes

[4] Anderson, J. T., Duclos-Cianci, G., & Poulin, D. (2014). Fault-tolerant conversion between the steane and reed-muller quantum codes. Physical review letters, 113(8), 080501.

[5] Quan, D. X., Zhu, L. L., Pei, C. X., & Sanders, B. C. (2018). Fault-tolerant conversion between adjacent Reed–Muller quantum codes based on gauge fixing. Journal of Physics A: Mathematical and Theoretical, 51(11), 115305.



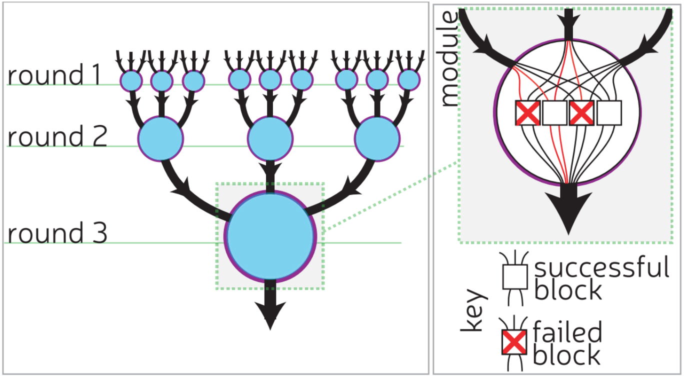
Subsystem Code Gauge Fixing

[6] Paetznick, A., & Reichardt, B. W. (2013). Universal fault-tolerant quantum computation with only transversal gates and error correction. Physical review letters, 111(9), 090505.

[7] Vuillot, C., Lao, L., Criger, B., Almud´ever, C. G., Bertels, K., & Terhal, B. M. (2019). Code deformation and lattice surgery are gauge fixing. New Journal of Physics, 21(3), 033028.

Magic State Distillation

- Magic state distillation implements a non-Clifford logical T gate.
- It is estimated to have a large resource overhead.



[8]O'Gorman, J., & Campbell, E. T. (2017). Quantum computation with realistic magic-state factories. Physical Review A, 95(3), 032338.

[9]Bombín, H. (2015). Gauge color codes: optimal transversal gates and gauge fixing in topological stabilizer

Conclusion

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2. Stabilizer theory is a mathematical framework for studying and designing quantum error-correcting codes.

3. Fault tolerance can be achieved by using transversal gates.

4. Surface codes are a family of topological stabilizer codes. Measurement-based protocols are used to realize different logical operations.

Thanks!