Quantum Error Correction : QSYS 23

1. No-cloning theorem: it is not allowed to copy an arbitrary state in quantum mechanics.

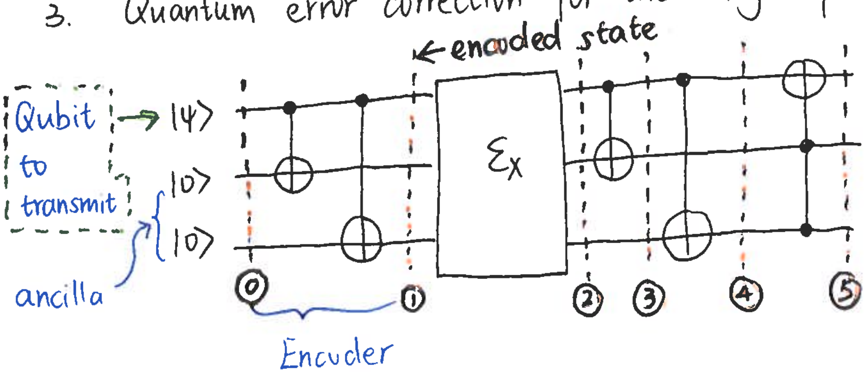
   Syndrome measurement is used for error correction.

2. Recall the 3-bit repetition code

   $0 \mapsto 000 \qquad 1 \mapsto 111$

   In qc, we can copy the basis states. But for the general (an arbitrary state) wave function $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, $\alpha, \beta \in \mathbb{C}$, we cannot copy.

3. Quantum error correction for the single-qubit Pauli X error : $X|a\rangle = |a \oplus 1\rangle$



Automatic correction

⓪ $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$|\psi\rangle|0\rangle|0\rangle = (\alpha|0\rangle + \beta|1\rangle)|0\rangle|0\rangle$

$\downarrow$ Encoder

① $\alpha|000\rangle + \beta|100\rangle \xrightarrow{CX_{12}} \alpha|000\rangle + \beta|110\rangle \xrightarrow{CX_{13}} \alpha|000\rangle + \beta|111\rangle$

$\downarrow \mathcal{E}\ (X_1, X_2, \text{or } X_3)$

② $\alpha|100\rangle + \beta|011\rangle$

$\downarrow CX_{12}$

③ $\alpha|110\rangle + \beta|011\rangle$

$\downarrow CX_{13}$

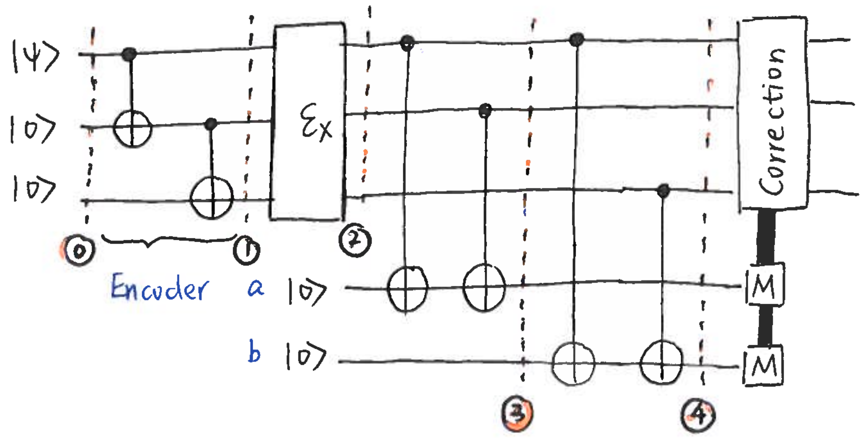
④ $\alpha|111\rangle + \beta|011\rangle$

$\downarrow CCX_{231}$

⑤ $\alpha|011\rangle + \beta|111\rangle = (\alpha|0\rangle + \beta|1\rangle)|1\rangle|1\rangle$

Exercise: What happens when the 2nd qubit is flipped?
What happens when the 3rd qubit is flipped?

Consider an equivalent QEC scheme:



⓪ $\alpha|000\rangle + \beta|100\rangle$

$\downarrow$ Encoder

① $\alpha|000\rangle + \beta|100\rangle \xrightarrow{CX_{12}} \alpha|000\rangle + \beta|110\rangle \xrightarrow{CX_{23}} \alpha|000\rangle + \beta|111\rangle$

$\downarrow$ $\mathcal{E}$ (X_1, X_2, or X_3)

② $\alpha|100\rangle + \beta|011\rangle$

③ $\alpha|100\rangle|0\rangle_a + \beta|011\rangle|0\rangle_a \xrightarrow{CX_{1a}} \alpha|100\rangle|1\rangle_a + \beta|011\rangle|0\rangle_a \xrightarrow{CX_{2a}} \alpha|100\rangle|1\rangle_a + \beta|011\rangle|1\rangle_a$

$= (\alpha|100\rangle + \beta|011\rangle)|1\rangle_a$

④ $\alpha|100\rangle|0\rangle_b + \beta|011\rangle|0\rangle_b \xrightarrow{CX_{1b}} \alpha|100\rangle|1\rangle_b + \beta|011\rangle|0\rangle_b \xrightarrow{CX_{3b}} \alpha|100\rangle|1\rangle_b + \beta|011\rangle|1\rangle_b$

$= (\alpha|100\rangle + \beta|011\rangle)|1\rangle_b$

$\downarrow$ Syndrome Measurement

"11" Means that the first bit is flipped.

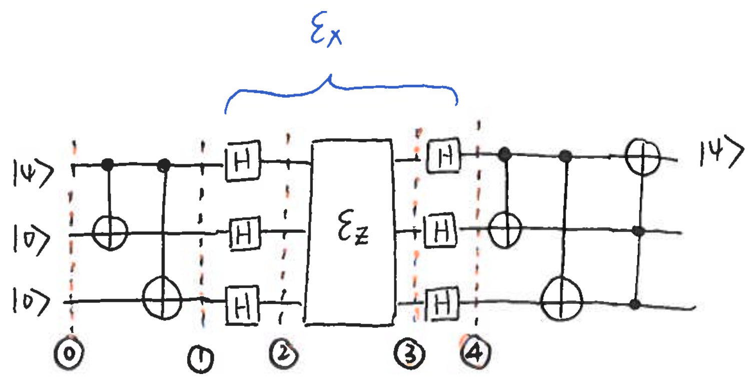
Exercise: What happens when the 2nd qubit is flipped?

What happens when no qubit is flipped?

Can you derive the syndrome table?

4. Quantum error correction for the single-qubit $\underbrace{\text{Pauli } Z}_{\text{or phase-flip}}$ error: $Z|a\rangle = (-1)^a |a\rangle$

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$



$HZH = X$

$H|0\rangle = |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ $\quad H|1\rangle = |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

$Z|+\rangle = |-\rangle$ $\quad Z|-\rangle = |+\rangle$

Exercise: Could you justify why the above QEC scheme corrects one Pauli Z error?

0 $\alpha|000\rangle + \beta|100\rangle$

$\downarrow$ Encoder

1 $\alpha|000\rangle + \beta|111\rangle$

$\downarrow$ $H \otimes H \otimes H$

2 $\alpha|{+}{+}{+}\rangle + \beta|{-}{-}{-}\rangle$

$\downarrow$ $\mathcal{E}_Z$ (Z_1 or Z_2 or Z_3)

3 $\alpha|{-}{+}{+}\rangle + \beta|{+}{-}{-}\rangle$

$\downarrow$ $H \otimes H \otimes H$

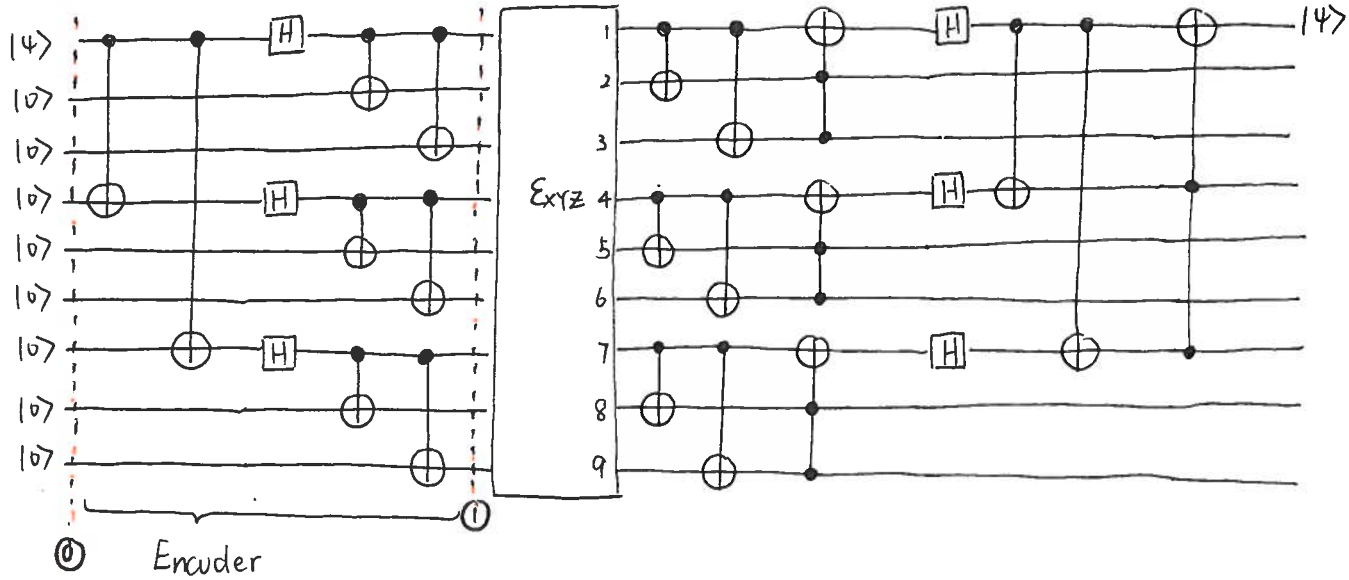
4 $\alpha|100\rangle + \beta|011\rangle$

Exercise: What happens when the 2nd qubit is phase-flipped?

What happens when no qubit is phase-flipped?

Can you derive the syndrome table?

5. Correct both the bit-flip and phase-flip error : Shor's 9-qubit code.



$$|0\rangle \mapsto \frac{1}{2\sqrt{2}}(|000\rangle+|111\rangle)\otimes(|000\rangle+|111\rangle)\otimes(|000\rangle+|111\rangle) = \frac{1}{2\sqrt{2}}(|000\rangle+|111\rangle)^{\otimes 3} =: |0\rangle_S$$

$$|1\rangle \mapsto \frac{1}{2\sqrt{2}}(|000\rangle-|111\rangle)\otimes(|000\rangle-|111\rangle)\otimes(|000\rangle-|111\rangle) = \frac{1}{2\sqrt{2}}(|000\rangle-|111\rangle)^{\otimes 3} =: |1\rangle_S$$

Exercise: What is the state at time slice 0 ?

What is the state at time slice 1 ?

$$|\psi\rangle = \alpha|0\rangle+\beta|1\rangle \xrightarrow{\text{encode}} \alpha|0\rangle_S + \beta|1\rangle_S$$ (encoded state)

Exercise: What happens to the encoded state after X_1 ?

What happens to the encoded state after Z_1 ?

What happens to the encoded state after $X_1 Z_1$?

$$\alpha|0\rangle_S+\beta|1\rangle_S \xmapsto{X_1} \frac{1}{2\sqrt{2}}\left[\alpha(|100\rangle+|011\rangle)\otimes(|000\rangle+|111\rangle)^{\otimes 2} + \beta(|100\rangle-|011\rangle)\otimes(|000\rangle-|111\rangle)^{\otimes 2}\right]$$

$$\alpha|0\rangle_S+\beta|1\rangle_S \xmapsto{Z_1} \frac{1}{2\sqrt{2}}\left[\alpha(|000\rangle-|111\rangle)\otimes(|000\rangle+|111\rangle)^{\otimes 2} + \beta(|000\rangle+|111\rangle)\otimes(|000\rangle-|111\rangle)^{\otimes 2}\right]$$

$$\alpha|0\rangle_S+\beta|1\rangle_S \xmapsto{X_1 Z_1} \frac{1}{2\sqrt{2}}\left[\alpha(|011\rangle-|100\rangle)\otimes(|000\rangle+|111\rangle)^{\otimes 2} - \beta(|100\rangle+|011\rangle)\otimes(|000\rangle-|111\rangle)^{\otimes 2}\right]$$