GRAPHICAL CSS CODE TRANSFORMATION USING ZX CALCULUS

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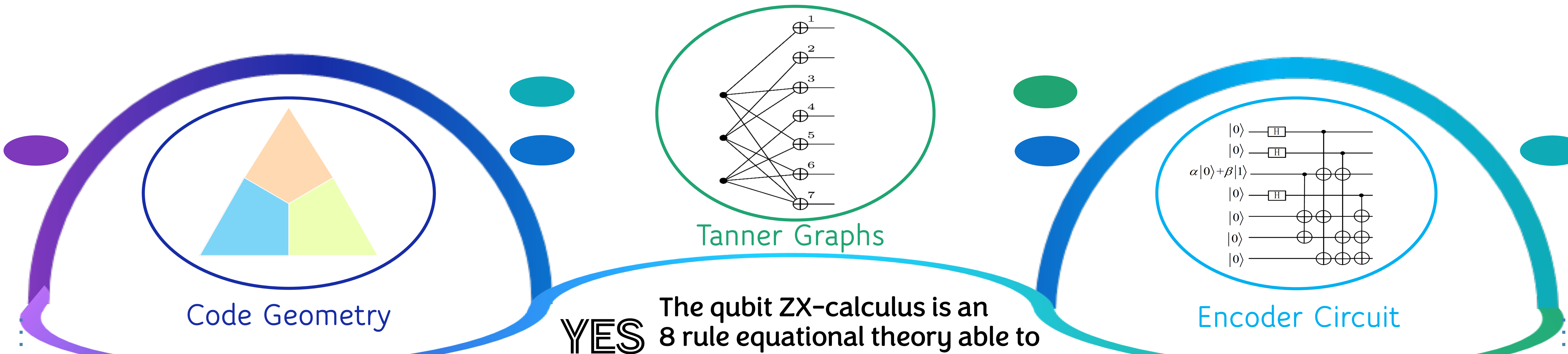
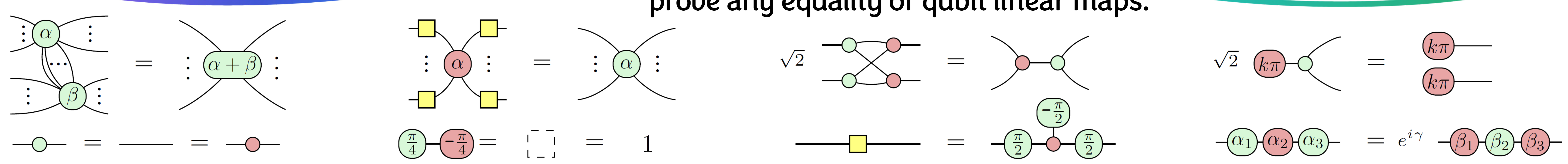
Can we unify the representation of CSS code in one graphical language?

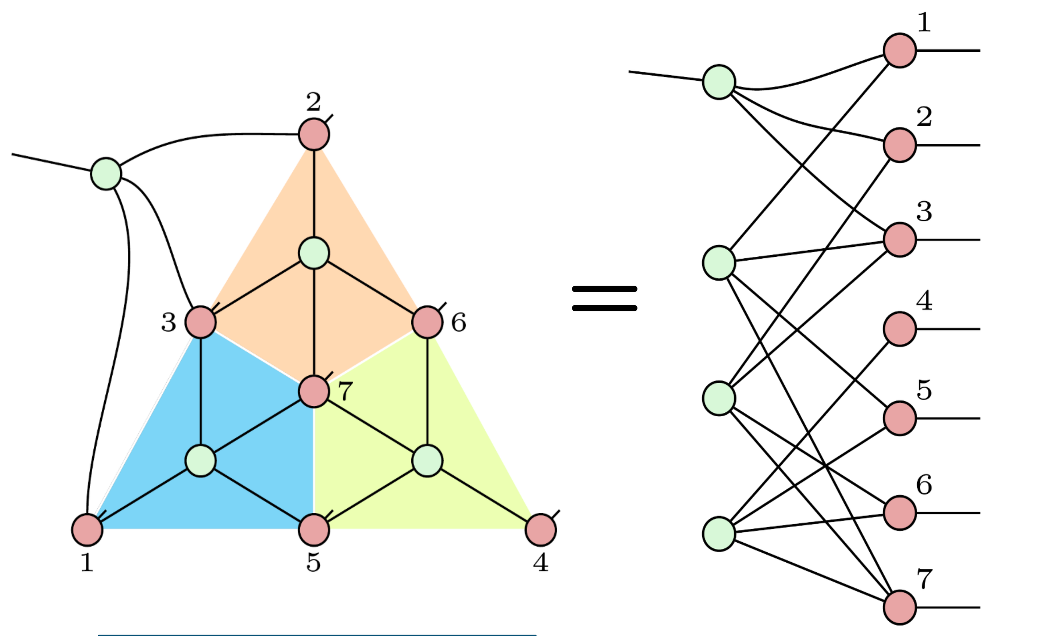


YES The qubit ZX-calculus is an 8 rule equational theory able to prove any equality of qubit linear maps.



where $m\{ \alpha \}^n := |0\rangle^{\otimes n}\langle 0|^{\otimes m} + e^{i\alpha}|1\rangle^{\otimes n}\langle 1|^{\otimes m}$ and $m\{ \alpha \}^n := |+\rangle^{\otimes n}\langle +|^{\otimes m} + e^{i\alpha}|-\rangle^{\otimes n}\langle -|^{\otimes m}$

Any CSS encoder has a phase-free ZX normal form which represents both the Tanner graph and geometry.

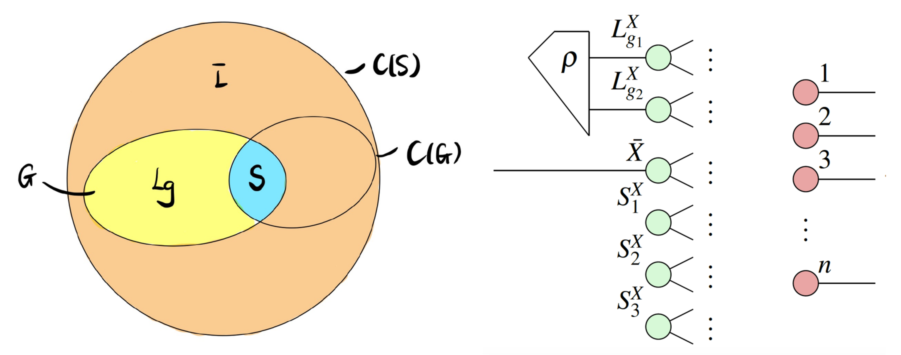


The 7-qubit Steane code | The ZX normal form

Definition: CSS Subsystem Codes

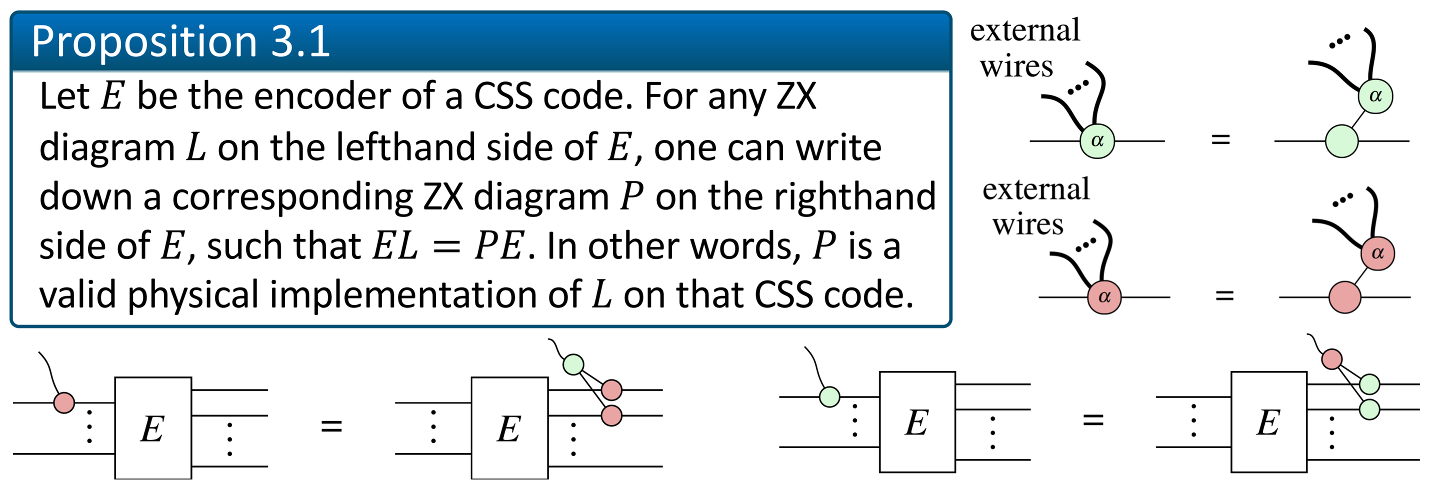
Let G be an arbitrary subgroup of the Pauli group $\mathcal{P}$. A subsystem code defined by G has a group S of stabilizers and a set L_g of gauge operators, where $S = C(G) \cap G$, $C(G) = \{P \in \mathcal{P}: PM = MP, \forall M \in G\}$, $L_g = G \backslash S$.

Subsystem codes are stabilizer codes where some of the logical qubits are not used for information storage and processing. These logical qubits are called gauge qubits.



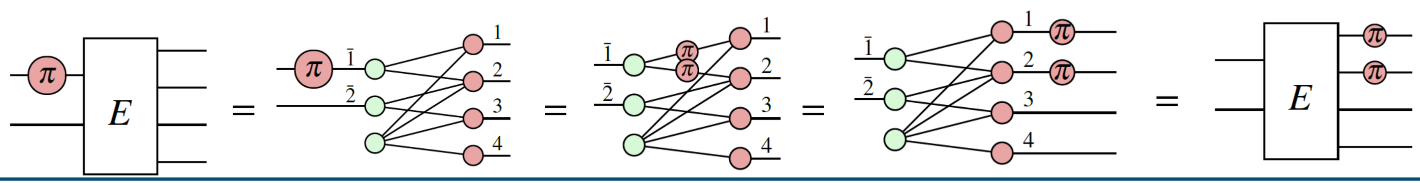
Proposition 3.1

Let E be the encoder of a CSS code. For any ZX diagram L on the lefthand side of E, one can write down a corresponding ZX diagram P on the righthand side of E, such that $EL = PE$. In other words, P is a valid physical implementation of L on that CSS code.



In any CSS code, all $\overline{X_i}$ and $\overline{Z_i}$ must be multi-qubit Pauli operators

Example: For the $[[4,2,2]]$ code, $\overline{X_1} = X_1X_2$



Graphical Transformations between CSS Codes

Case Study: The [[15, 4, 3]] Subsystem Quantum Reed-Muller (SQRM) Code

SQRM is defined by the gauge group:

$$G = \langle N^X, N^Z, H^X, H^Z, T^Z \rangle.$$

The associated stabilizer group, gauge group, and logical operators are:

$$\mathbb{S}_S = \langle N^X, N^Z, H^Z \rangle_{11}, \quad L_g = \langle H^X, T^Z \rangle_6, \quad \bar{L} = \langle \bar{X}, \bar{Z} \rangle_2$$

$$M = \begin{bmatrix} 1&0&1&0&1&0&1 \\ 0&1&1&0&0&1&1 \\ 0&0&0&1&1&1&1 \end{bmatrix}_{3\times 7} \quad N = \begin{bmatrix} M & 0 & M \\ \mathbf{0} & 1 & \mathbf{1} \end{bmatrix}_{4\times 15} \quad H = \begin{bmatrix} M & \mathbf{0} \end{bmatrix}_{3\times 15} \quad T = \begin{bmatrix} 1&0&1&0&0&0&0&0&1&0&1&0&0&0&0 \\ 0&1&1&0&0&0&0&0&0&1&1&0&0&0&0 \\ 0&0&1&0&0&0&1&0&0&0&1&0&0&0&1 \end{bmatrix}_{3\times 15}$$

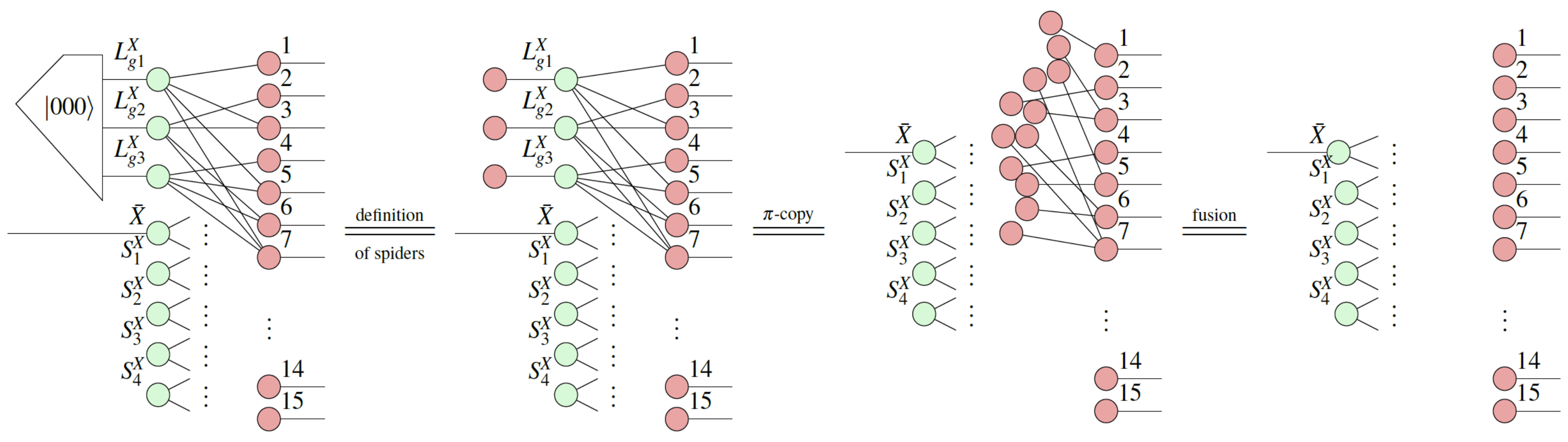
Gauge Fixing SQRM Converts between Quantum Reed-Muller (QRM) code and the extended QRM

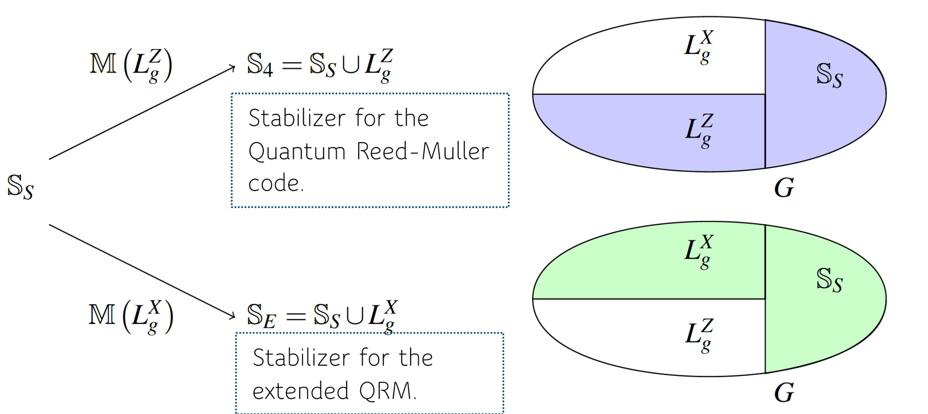
Step 1: Measure a commuting subset of gauge operators. E.g., measure three X-type gauge operators $L^X_{g_i}$ and obtain the corresponding outcomes $k_1, k_2, k_3 \in \{0,1\}$:

$$L^X_{g_i}|\bar{\psi}\rangle = (-1)^{k_i}|\bar{\psi}\rangle.$$

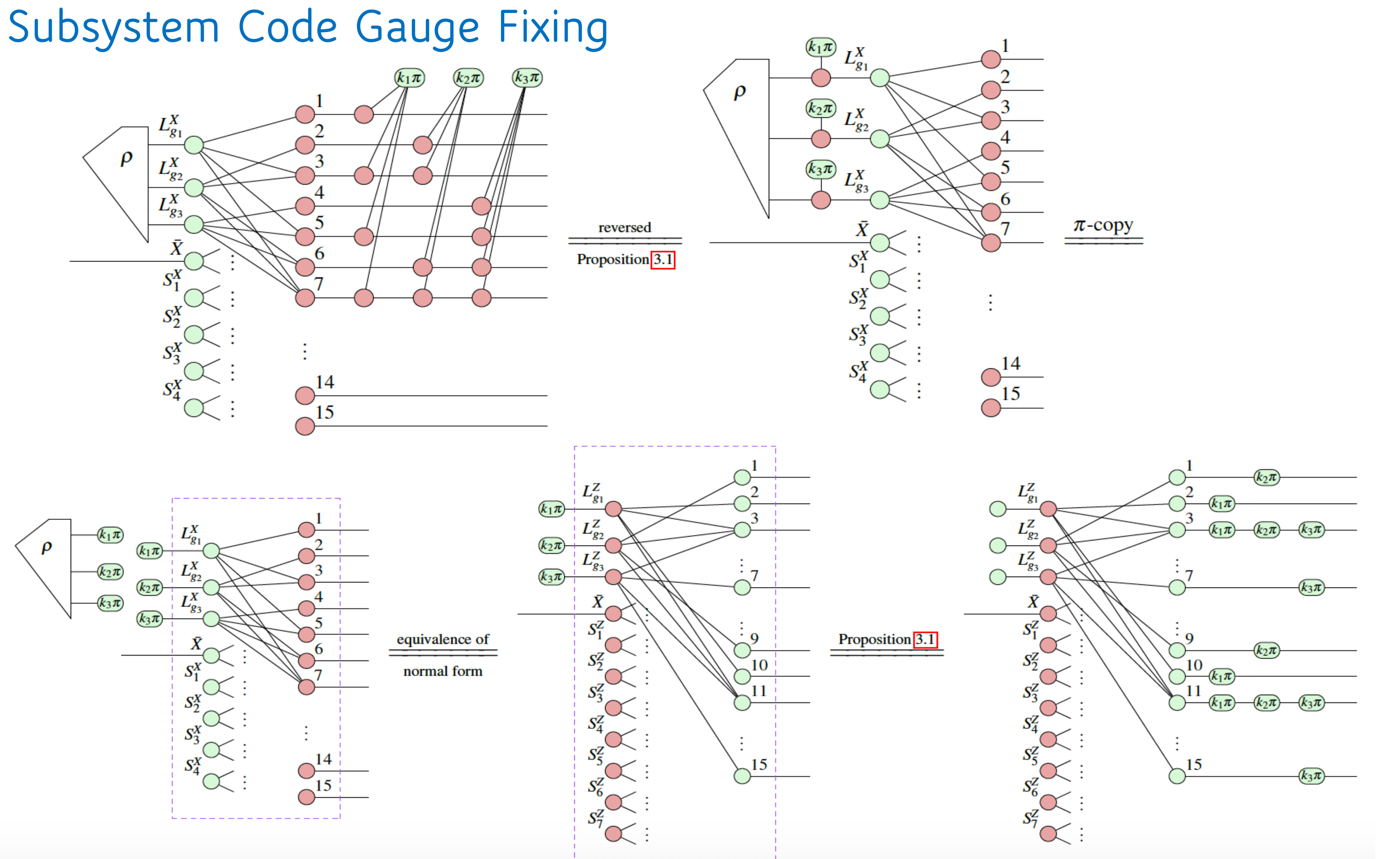
Step 2: When $k_i = 1$, the state of gauge qubit i is projected to the wrong state $|{=}\rangle$. Applying the recovery operator $L^Z_{g_i}$ corrects the collapsed state:

$$L^Z_{g_i}|{=}\rangle = |{\mp}\rangle.$$



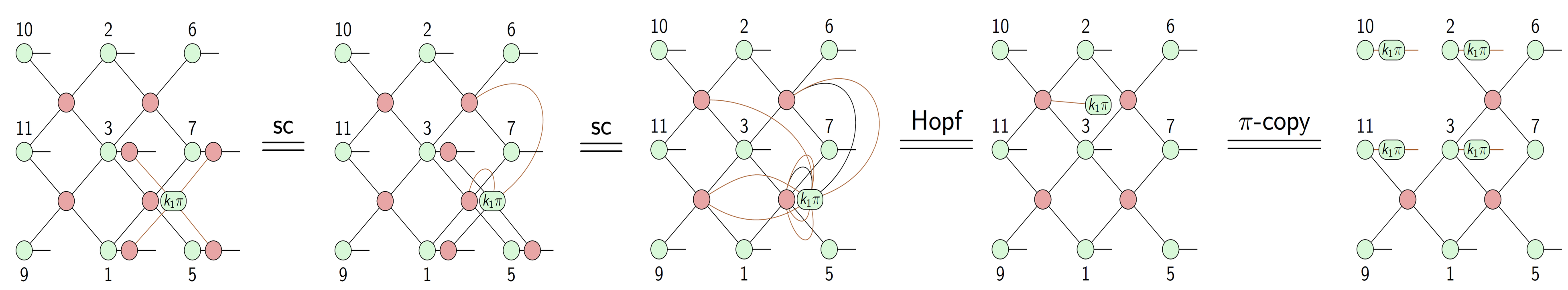


Subsystem Code Gauge Fixing

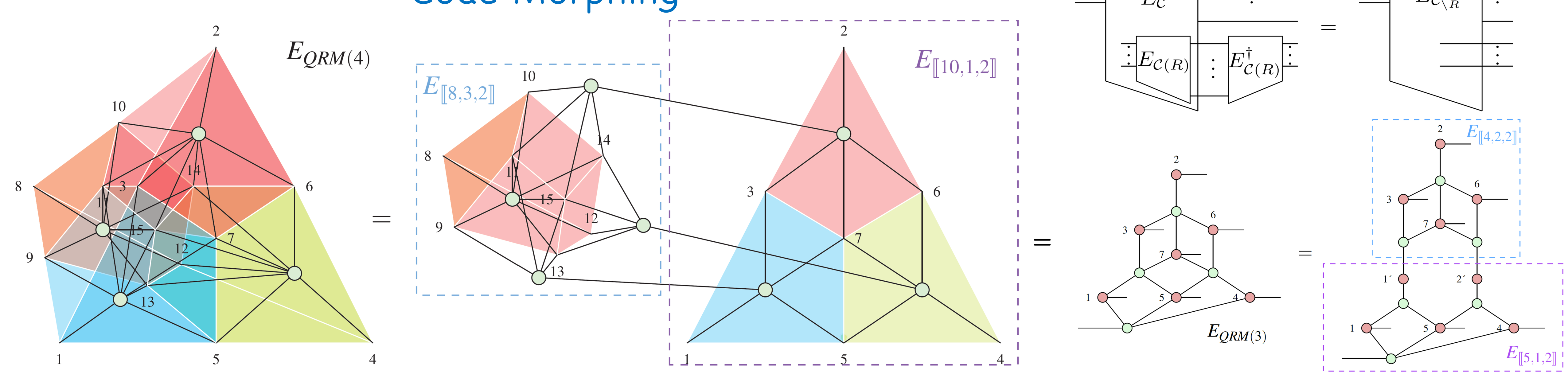


Syndrome-determined Recovery Operation

Measuring $L^X_{g_i}$ adds these operators into the stabilizer group and removes stabilizers $L^Z_{g_i}$. One can readily read off the fixing operations from the graphical derivation.



Code Morphing



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