PROBABILITY & GAMES

MATH CIRCLE

WHAT IS AN EVENT?

- In probability theory, an **event** is a set of outcomes of an experiment (a subset of the sample space) to which a probability is assigned.
- An **experiment** is a repeatable procedure with a set of possible results.
- When the sample space is finite, any subset of the sample space is an event.
- Now, let's define some events for given experiments!

EXPERIMENTS & EVENTS

- **Experiment I:** There are five horses (labeled 1 ~ 5) participating in a horse race, and we assume that each horse has an equal chance to win the race.
- **Experiment II:** Tossing two dice.
- **Experiment III:** Drawing two cards from a deck.

Describe an event for each experiment.

WHAT IS PROBABILITY?

- Many events cannot be predicted with complete certainty. The best we can say is how **likely** they are to happen using probability.
- Tossing a fair coin
  - When a coin is tossed, there are two possible outcomes:
    - Heads (H) or
    - Tails (T)
  - We say that the probability of the coin landing on H is ½. And the probability of the coin landing on T is ½.

PROBABILITY

- How can we calculate the probability of an event?

$$Probability\ of\ an\ event\ happening = \frac{number\ of\ ways\ it\ can\ happen}{total\ number\ of\ outcomes}$$

- Calculate the probability of the following events from the above experiments.
    1. Horse 2 wins the race (there are in total five horses in the race).
    2. You get two even numbers from tossing two dice.
    3. You draw two spades from the deck.

FUNDAMENTAL COUNTING PRINCIPLE

WHEN THERE ARE **M** WAYS TO DO SOMETHING, AND **N** WAYS TO DO ANOTHER THING, THEN THERE ARE **M*N** POSSIBLE OUTCOMES.

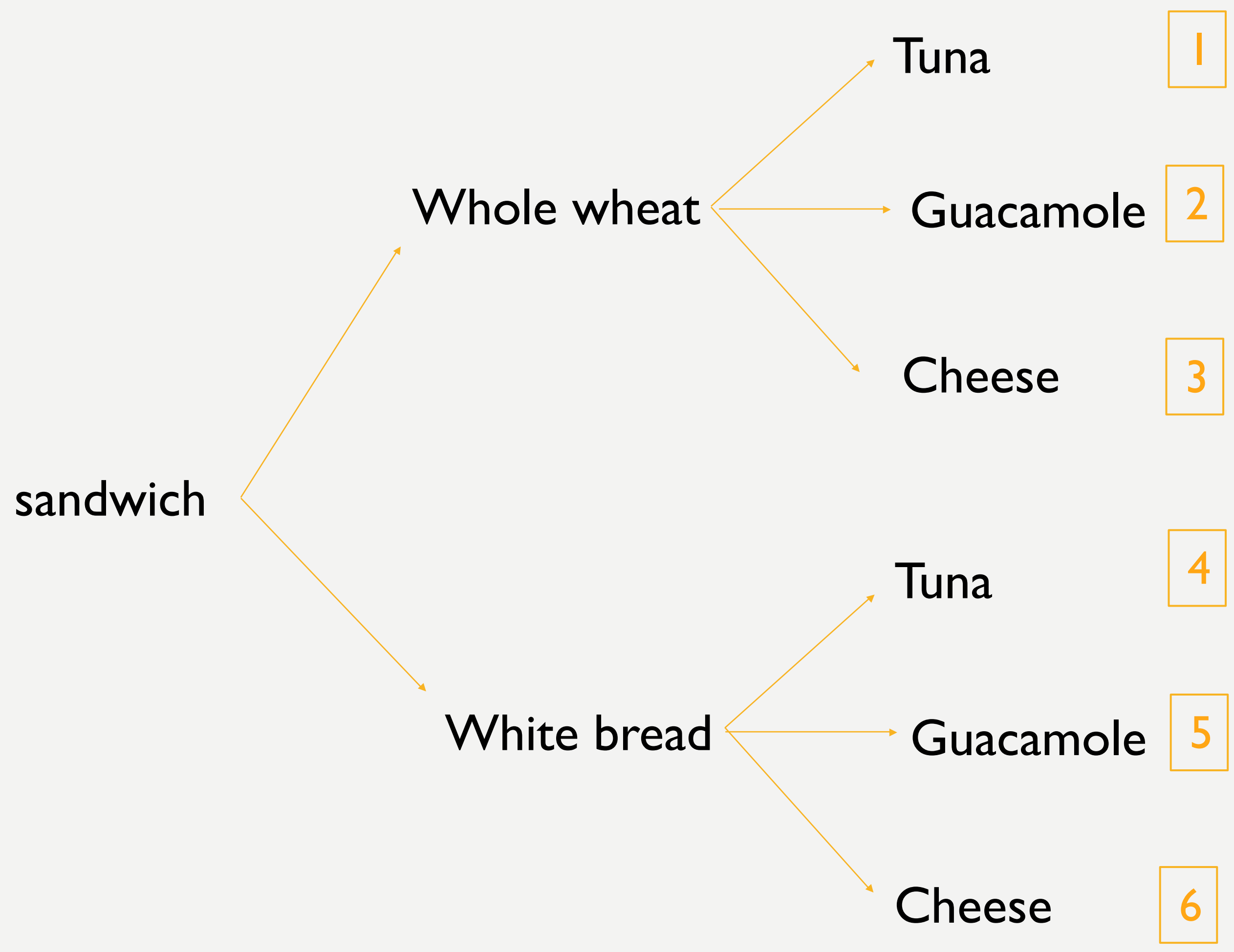
EXAMPLE

I bring lunch to school everyday.

- At home, I get two types of bread: whole wheat bread and white bread.
- I also have three types of fillings that I'd like to use: tuna, guacamole, and cheese.

How many different combinations can I make?

What if I want to add one of the four types of fruits into my lunch box: apple, banana, orange, or watermelon?



DECISION TREE

How many different combinations of breads and fillings can I have?

There are 6 possible outcomes with breads and filling.

$$2 * 3 = 6$$

There are 24 possible outcomes with breads, filling, and fruits.

$$2 * 3 * 4 = 24$$

MENU TOSS-UP

- How many combinations can a customer make from the lunch menu with one starter, one house special, and one dessert?
- If each combination is equally likely to be picked by a random customer, what is the probability that a customer orders a homemade soup, a Medieval beef & Guinness stew, and an apple pie?

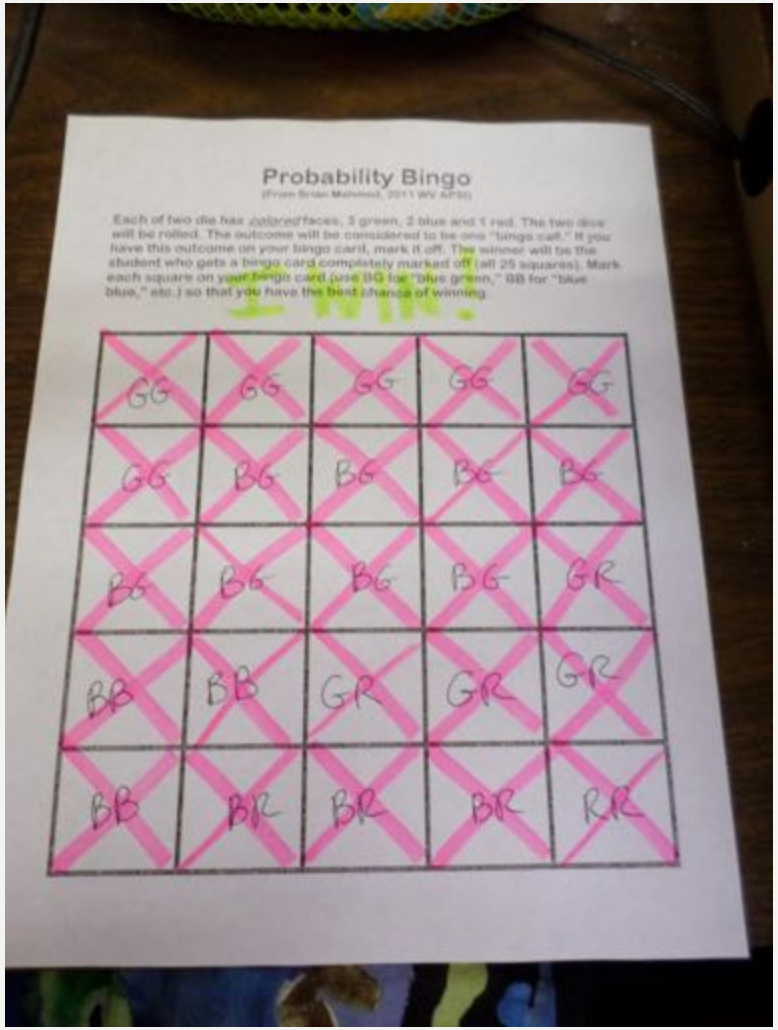
PROBABILITY BINGO

- Each of two die has colored faces: 3 green, 2 blue and 1 red. The two dice will be rolled. The outcome will be considered to be one “bingo call.” If you have this outcome on your bingo card, mark it off.

WHO WILL BE THE FIRST TO WIN?

PROBABILITY BINGO

- Find yourself a bingo partner.
- You can fill in your bingo card however you want to help you be the first to cross all cells in the bingo card.
- You and your partner take turns rolling the two dice. With each roll, if the combination exists in your bingo card and had not been marked before, mark it off.
- The winner will be the person who gets a bingo card completely marked off (all 25 squares). Mark each square on your bingo card (use BG for “blue green,” BB for “blue blue,” etc.) so that you have the best chance of winning.



COLORED DICE PROBABILITY

	G	G	G	B	B	R
G	GG	GG	GG	BG	BG	GR
G	GG	GG	GG	BG	BG	GR
G	GG	GG	GG	BG	BG	GR
B	BG	BG	BG	BB	BB	BR
B	BG	BG	BG	BB	BB	BR
R	GR	GR	GR	BR	BR	RR

ANY TRICKS TO WIN THE GAME?

HINTS:

1. HOW MANY POSSIBLE OUTCOMES ARE HERE?
2. ARE THEY EQUALLY LIKELY?
3. MAKE YOURSELF A TABLE TO DEMONSTRATE ALL POSSIBLE OUTCOMES AND THE CORRESPONDING PROBABILITIES.

COLORED DICE PROBABILITY

	G	G	G	B	B	R
G	GG	GG	GG	BG	BG	GR
G	GG	GG	GG	BG	BG	GR
G	GG	GG	GG	BG	BG	GR
B	BG	BG	BG	BB	BB	BR
B	BG	BG	BG	BB	BB	BR
R	GR	GR	GR	BR	BR	RR

$P(BG) = \frac{12}{36} = .333$

$P(GG) = \frac{9}{36} = .250$

$P(GR) = \frac{6}{36} = .167$

$P(BB) = \frac{4}{36} = .111$

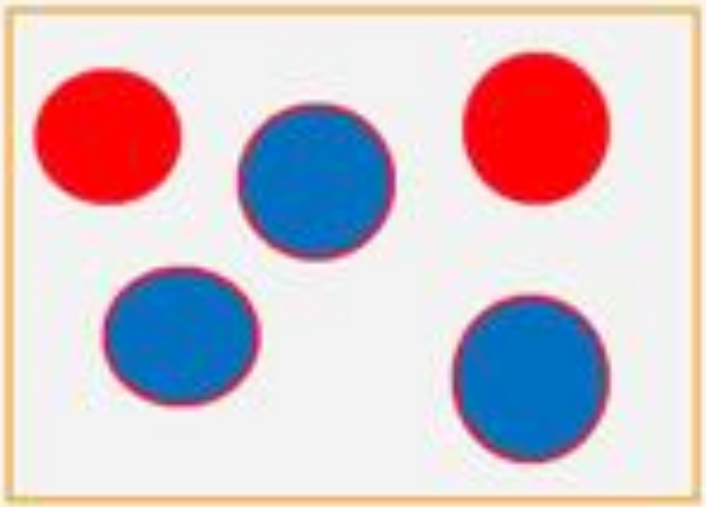
$P(BR) = \frac{4}{36} = .111$

$P(RR) = \frac{1}{36} = .028$

CONDITIONAL PROBABILITY& TREE DIAGRAM

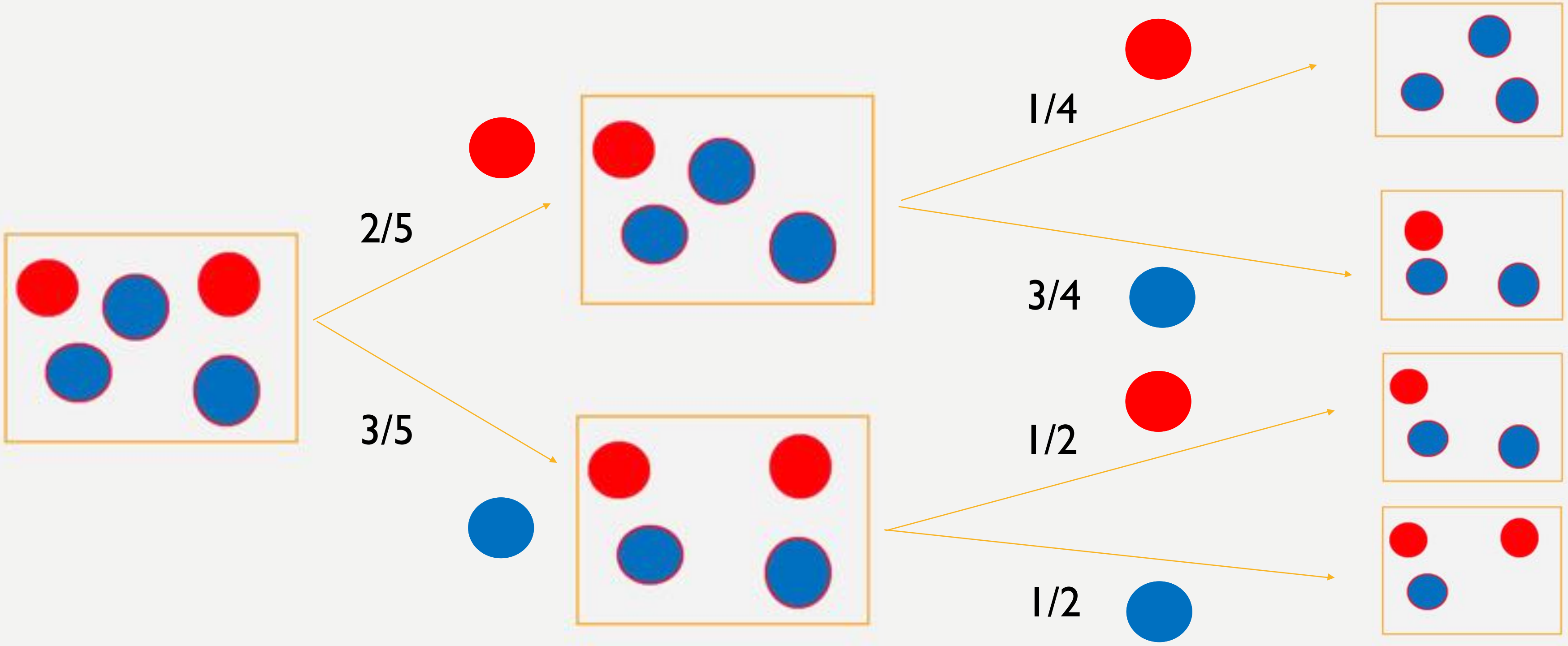
DEPENDENT & INDEPENDENT EVENTS

- Events can be independent if each event is not affected by any other events.
  - When tossing a coin, each toss is an independent event.
- Events are dependent if they can be affected by previous events.
  - Suppose there are 2 blue and 3 red marbles in a bag. What are the chances of getting a blue marble?
  - After taking one blue marble out, what are the chances of getting another blue marble?
  - Are the probability of these two events the same? Why or why not?



TREE DIAGRAM

- Tree diagrams display all the possible outcomes of an event. Each vertex (node) in a tree diagram represents a possible outcome. Each branch represents the probability that corresponds to the event. Tree diagrams can be used to find the number of possible outcomes and calculate the probability of possible outcomes.



INTERPRETING THE TREE DIAGRAM

- Pair discussion: form a group of two students and explain the meaning of each probability in the diagram above.
- Now let's answer the following questions.
    1. What are the chances of drawing two blue marbles?
    2. After drawing one blue marble, what is the chance of drawing one red marble?

CONDITIONAL PROBABILITY

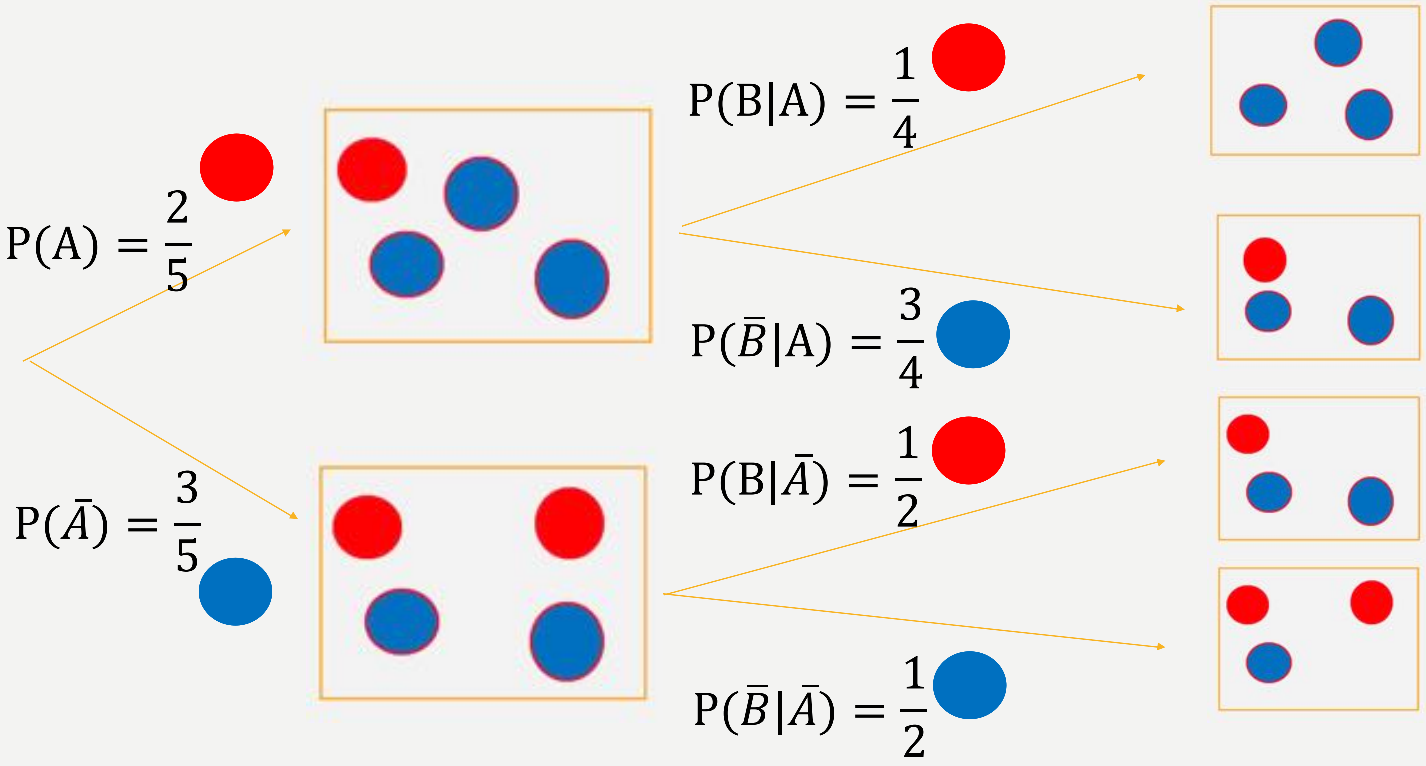
- The conditional probability of an event B given event A is the probability that the event B will occur given the knowledge that an event A has already occurred. This probability is written P(B|A).
- In the case where events A and B are independent (where event A does not affect the probability of event B), the conditional probability of event B given event A is simply the probability of event B; that is, P(B).
- In the case where events A and B are dependent

$$P(B|A) = \frac{P(A\ and\ B)}{P(A)}$$

- There is another way to understand conditional probability: counting elements in the set.
- In the next few slides, we will give a detailed description about conditional probability in the marble example.

NOTATION

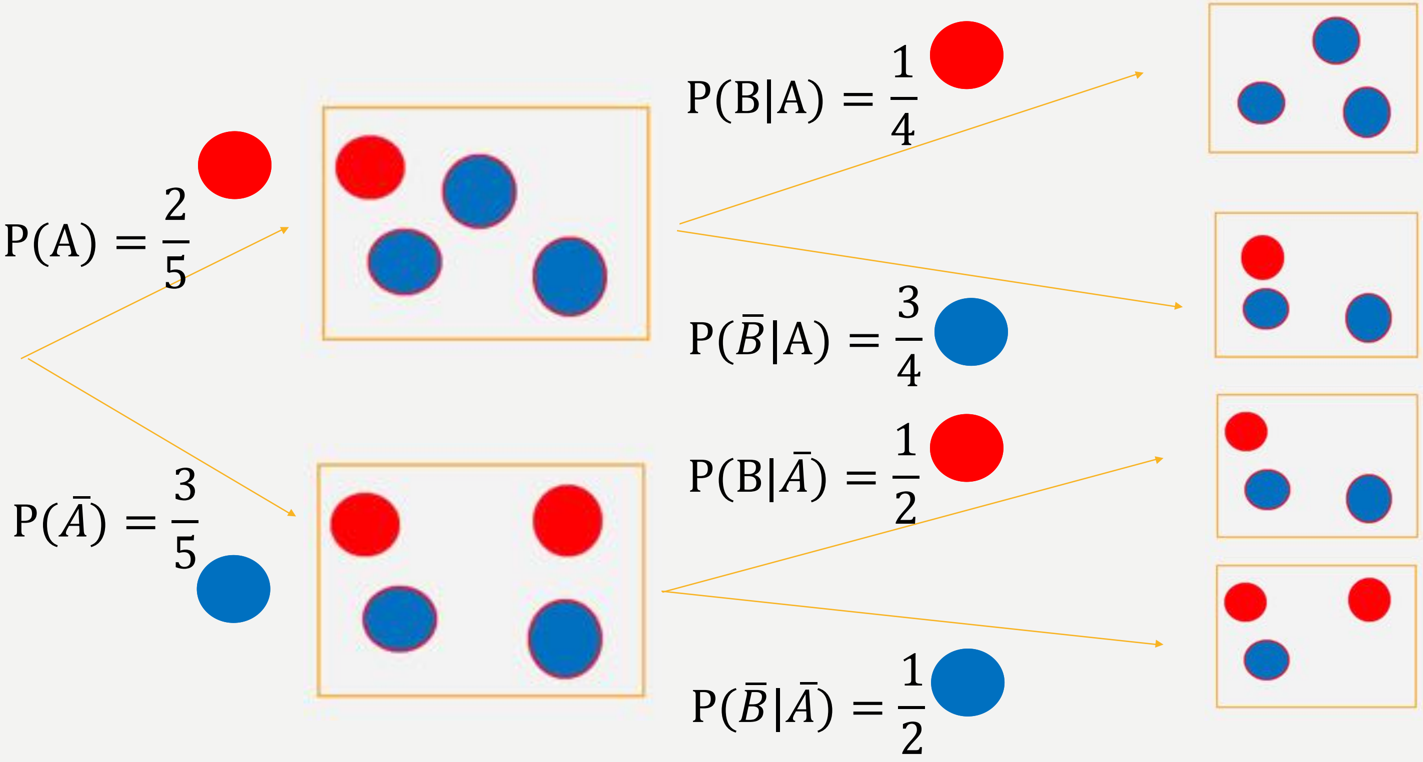
- Let A denote the event of getting a red ball from the box in the first round. Then P(A) = $\frac{2}{5}$
- Let $\bar{A}$ be an event of getting a blue ball from the box in the first round. $\mathrm{P}(\bar{A})$ means the "probability of *not* event A". Then $\mathrm{P}(\bar{A}) = \frac{3}{5}$
- Let B denote the event of getting a red ball from the box in the second round. $\mathrm{P}(B|A) = \frac{1}{4}$
- Then $\bar{B}$ denotes the event of getting a blue ball from the box in the second round.
- Then $\mathrm{P}(\bar{B}|A) = \frac{3}{4}$



MORE QUESTIONS

- What is the probability of getting two blue balls?
- What is the probability of getting one blue ball and one red ball?
- What is the probability of getting two red balls?
- What is the probability of getting blue balls in the second round?
- What is the probability of getting red balls in the second round?
- ……

FINDING HIDDEN DATA



$$P(\text{A and B}) = P(B|A) * P(A)$$

$$P(\bar{B} \text{ and } A) = P(\bar{B}|A) * P(A)$$

$$P(B\ and\ \bar{A}) = P(B|\bar{A}) * P(\bar{A})$$

$$P(\bar{B}\ and\ \bar{A}) = P(\bar{B}|\bar{A}) * P(\bar{A})$$

THE CLIFF HANGER

From where he stands, one step toward the cliff would send the drunken man over the edge. He takes random steps, either toward or away from the cliff. At any step his probability of taking a step away is $\frac{2}{3}$, of a step toward the cliff $\frac{1}{3}$. What is his chance of escaping the cliff?

BAYES' THEOREM

$$P(A|B) = \frac{P(A)(B|A)}{P(B)}$$

- P(A|B): How likely A happens given that B happens
- P(B|A): How likely B happens given that A happens
- P(A and B): How likely A and B both happen

$$P(A\ and\ B) = P(A|B)P(B) = P(B|A)P(A)$$

EXAMPLE: PICNIC DAY

- Your class is planning a picnic day, but this morning is cloudy.
- Oh no! 50% of all rainy days start off cloudy!
- But cloudy mornings are common (about 40% of days start cloudy).
- This is usually a dry month (only 3 of 30 days tend to be rainy).

WHAT IS THE CHANCE OF RAIN DURING THE DAY?

THINK LIKE A BAYESIAN STATISTICIAN!

- Let Rainy denote raining during the day, and Cloudy denote cloudy morning.

- Then $P(Rainy) = \frac{3}{30} = \frac{1}{10}, P(Cloudy) = \frac{40}{100} = \frac{2}{5}$
- Then $P(Cloudy|Rainy) = \frac{50}{100} = \frac{1}{2}$

- Now we want to know $P(Cloudy|Rain)$
- Using Bayes' Theorem, we have

$$P(Rainy|Cloudy) = \frac{P(Cloudy|Rainy)P(Rainy)}{P(Cloudy)} = \frac{\frac{1}{2} * \frac{1}{10}}{\frac{2}{5}} = \frac{0.5 * 0.1}{0.4} = 0.125$$

THE MONTY HALL PROBLEM

Monty Hall was the host of a gameshow called *Lets Make a Deal!* On one episode, Monty presented a challenge to a lucky contestant: 3 doors, behind one of which was the prize; behind the other two doors: goats.

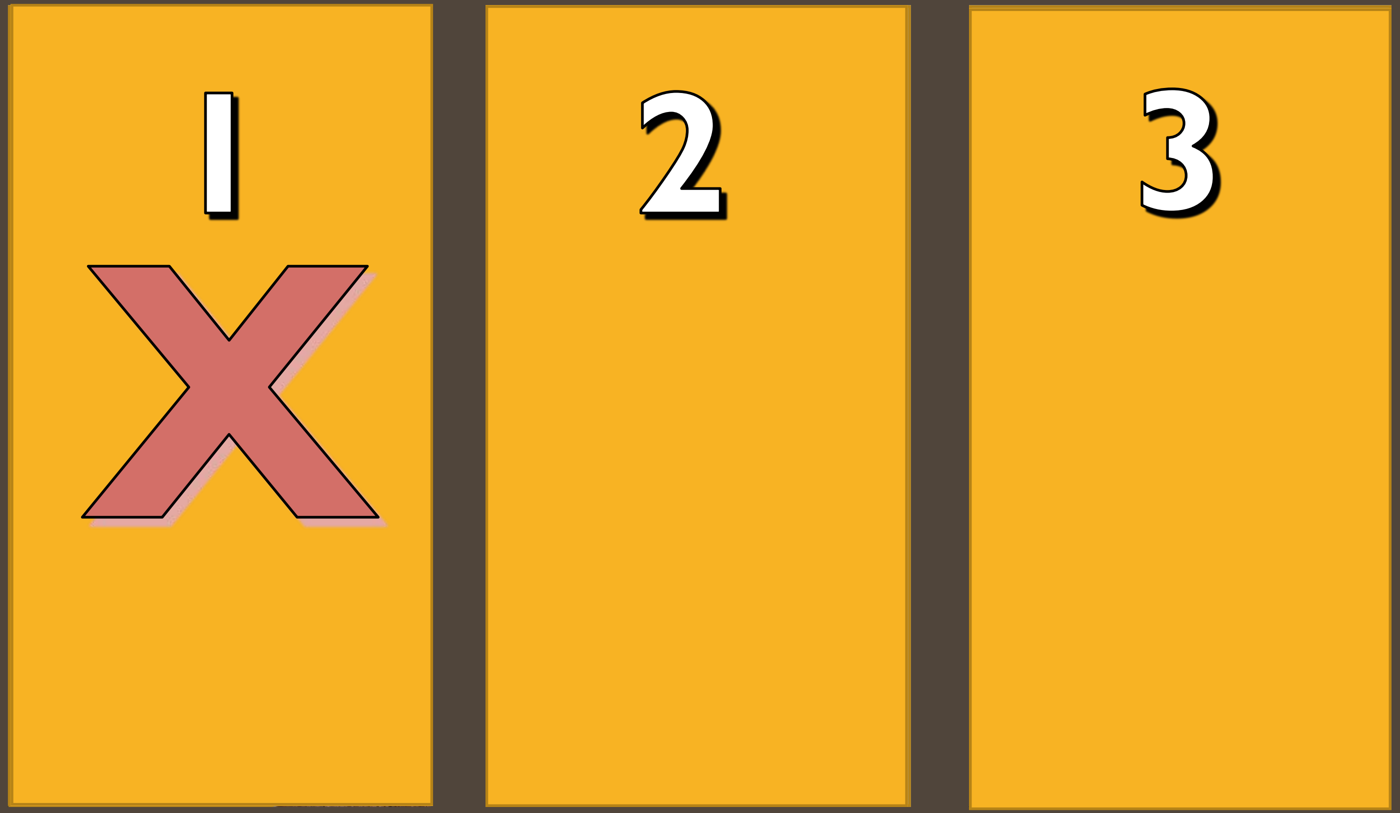
The contestant was instructed to pick a door, but not to open it yet.

Once the contestant had chosen a door, Monty opened one of the doors the contestant didn't choose to reveal a goat.

The contestant is then presented with a choice: stay with their choice of door, or switch to the other door.

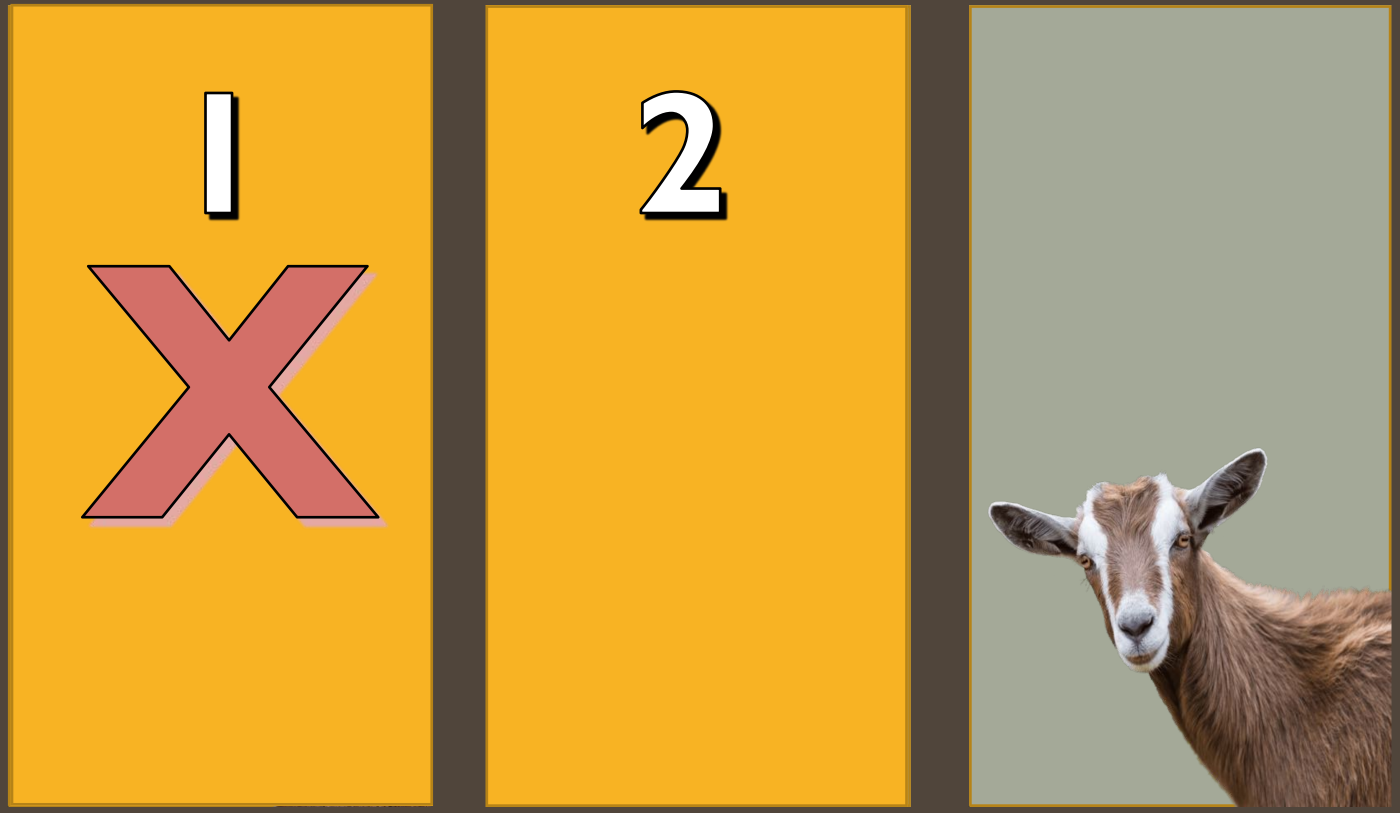
HOW TO PLAY

1. Contestant chooses a door
2. Monty reveals a goat



HOW TO PLAY

1. Contestant chooses a door
2. Monty reveals a goat
3. Contestant opens his door



HOW TO PLAY

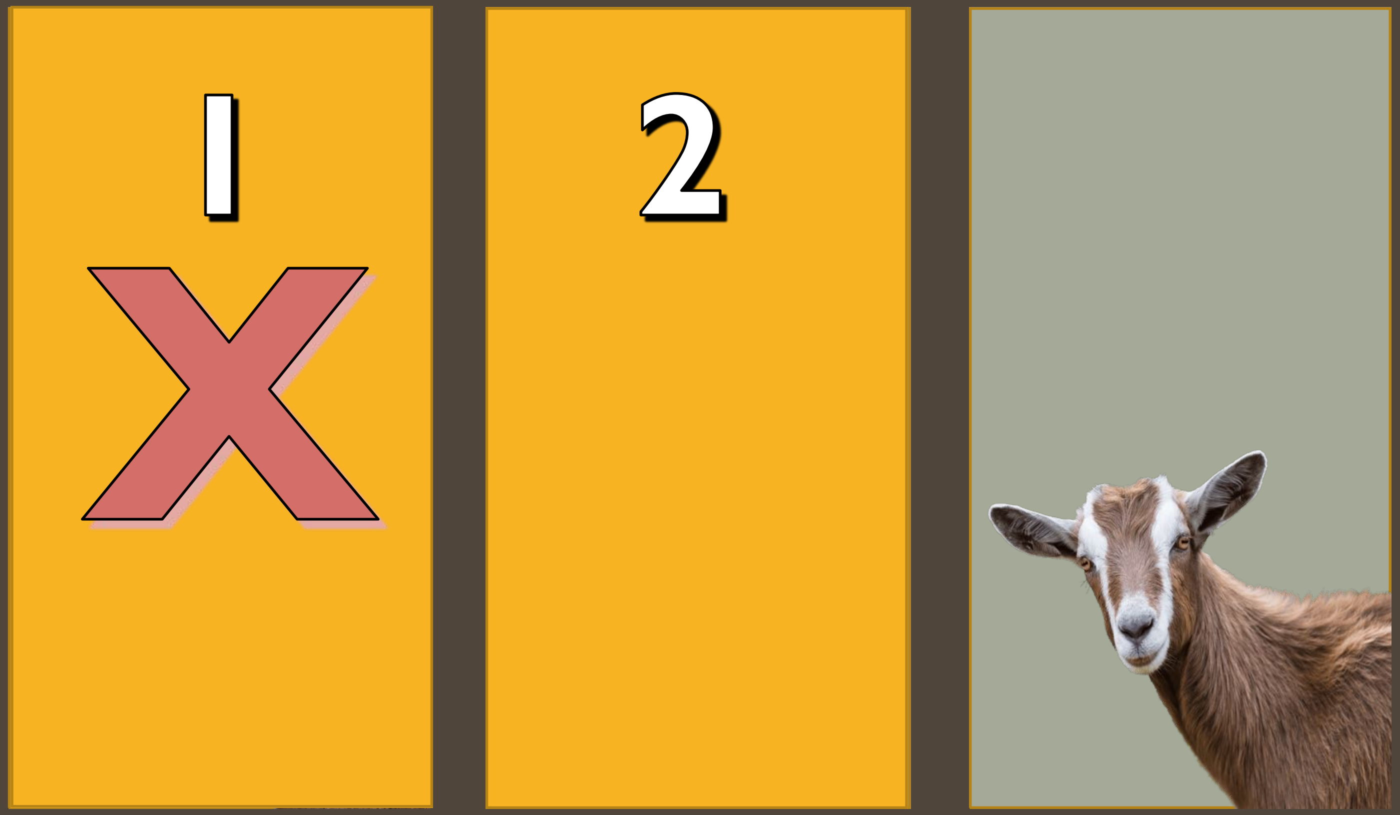
1. Contestant chooses a door

2. Monty reveals a goat

3. Contestant opens his door

OR

3. Contestant switches choice of door



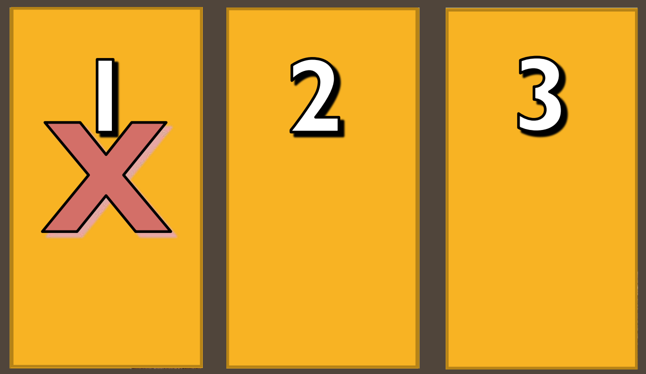
WHAT'S THE BEST STRATEGY?

DOES IT MAKE A DIFFERENCE IF THE PARTICIPANT SWITCHES DOORS?

WHAT'S THE BEST STRATEGY?

In groups of two (or three), try the game out yourselves!

Decide on a person to be Monty and a person to be the Contestant. Play 12 trials where the contestant stays with their original choice.
Then, switch roles. Play 12 trials where the contestant switches doors. Record your results in a table.



MORE PROBABILITY PUZZLES

THE FLIPPANT JUROR

A three-man jury has two members each of whom independently has probability p of making the correct decision and a third member who flips a coin for each decision (majority rules). A one-man jury has probability p of making the correct decision. Which jury has the better probability of making the correct decision?

THE BIRTHDAY PROBLEM

What is the least number of people required in a room so that that the probability that two of them share a birthday exceeds a half?

Assume all birthdays are equally likely and disregard leap years.

DOWRY PROBLEM

The king, to test a candidate for the position of wise man, offers him a chance to marry the young lady in the court with the largest dowry. The amounts of the dowries are written on slips of paper and mixed. A slip is drawn at random and the wise man must decide whether that is the largest dowry or not. If he decides it is, he gets the lady and her dowry if he is correct; otherwise he gets nothing. If he decides against the amount written on the first slip, he must choose or refuse the next slip, and so on until he chooses one or else the slips are exhausted. In all, 100 attractive young ladies participate, each with a different dowry. How should the wise man make his decision?

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